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Abstract

We provide an algorithmic approach to the construction of point transformations for scalar ordinary differential equations that admit three-dimensional symmetry algebras which lead to their respective canonical forms.

Keywords: ODEs, three-dimensional symmetry algebras, point transformations, canonical forms.

1 Introduction

This is a contribution to the algorithmic theory of differential equations in the sense of Schwarz [6]. In the first five sections we provide results needed to construct algorithms, which are illustrated in detail in section 6 of the paper.

Not all the three-dimensional algebras have invariant differential equations of second-order in the sense that they are the full symmetry algebra of the equation. For this reason, we have given higher-order invariant equations for such types of three-dimensional algebras.

Although all the results of this paper were independently arrived at, the main ideas are already in Lie [4]. The justification for placing this in the public domain is its brevity, clarity and a uniform treatment of compact and non-compact algebras.

The realizations of two- and three-dimensional Lie algebras as vector fields in \mathbb{C}^2 is given in e.g., Ibragimov [7, page 163]. The details, as well as the invariant second-order ordinary differential equations (ODEs) are given in Lie [4, pages 479-542]. The two-dimensional algebras are essentially distinguished by their ranks. It is thus desirable to give a similar description of three-dimensional Lie algebras over the reals.

The main aim of this work is to give such a description and an algorithmic procedure that systematically utilizes the structural information more explicitly and which is programmable. This program reduces any given ODE that admits a three-dimensional algebra to its simplest form.

This is achieved by proving a version of the Lie-Bianchi classification in an algorithmic way and giving the realizations of the algebras as vector fields in \mathbb{R}^2 .

In every case, there is an invariantly defined two-dimensional abelian algebra - which is not always a subalgebra of the given algebra - and its rank determines the coordinates that reduces the equation to its simplest form.

As far as the form of the invariant differential equations is concerned, one can use the result in [8, pages 69-76] that reduces the computation of local joint invariants of any finite number of vector fields algorithmically to that of an abelian algebra of vector fields.

A few words regarding the importance of low-dimensional algebras for differential equations are in order.

Lie [3,4] obtained the complete explicit classification of scalar second-order ODEs that possess non-similar (not transformable into each other via point transformation) complex Lie algebras of dimension r , where $r = 0, \dots, 8$. He showed that the complex Lie algebra of vector fields acting in the plane admitted by a given second-order ODE can only be of dimensions 0, 1, 2, 3 or 8. He also proved that if a second-order equation admits an eight-dimensional algebra, it is linearizable by means of a point transformation and it is then equivalent to the simplest equation, viz. the free particle equation.

It is well-known that one- and two-dimensional algebras have identical structures over the reals as well as over the complex numbers. As a consequence, the Lie symmetry algebra classification of scalar second-order ODEs over the reals is precisely the same as that over the complex numbers for one- and two-dimensional Lie-algebras.

If a second-order equation admits a single generator of symmetry, then in general its order can be reduced by one [4]. Moreover, Lie [4] showed that scalar second-order equations possessing two generators of symmetry have four canonical forms. These are well-known now as the Lie canonical

forms for the vector fields and their representative second-order equations. Lie [4] also proved that the rank one algebras result in linearization of the associated second-order ODE.

The situation is different for three- and higher-dimensional Lie algebras as there are fewer complex than real algebras of dimension three or more. This arose in the Bianchi [2] classification of Lie algebras. Two of the complex Lie algebras of dimension three in the real plane each split up into two real non-isomorphic Lie algebras. Therefore, there are two more non-isomorphic real three-dimensional Lie algebras than complex algebras.

Due to the above considerations on Lie algebras in the real plane for higher dimensions, there are additional three-dimensional algebras of vector fields acting in the real plane than in the complex plane. These were deduced by Mahomed and Leach [5]. These yield additional non-similar scalar second-order equations that admit real Lie algebras [5].

In summary, this is a contribution to the algorithmic Lie theory of scalar ODEs in the sense of Schwarz [6]. Schwarz [6] utilized Janet bases in the representation of the determining equations of the symmetry generators. The main difference in our approach is to use canonical forms of the symmetry algebra in order to construct the requisite point transformations that bring a given ODE with known symmetry algebra to its canonical form.

The reader is referred to Ibragimov [7] for the background on Lie's theory of symmetries of differential equations.

2 The Lie-Bianchi Classification of three-dimensional solvable algebras

We begin with a formulation of the Lie-Bianchi [1,2] classification of three-dimensional Lie algebras.

Theorem 2.1. *Let G be a three-dimensional Lie algebra. If G is solvable, then G' is abelian. Moreover,*

(a) *if G' is two-dimensional, then the structure of G is completely determined by the eigenvalues and multiplicities of $\text{ad}(X)$ as a linear transformation of G' , where X is any representative of G/G' in G ,*

(b) *if G' is one-dimensional, then the structure of G is completely determined by the dimension of the centralizer of G' in G .*

Proof. In general, by Lie's theorem on complex solvable linear Lie algebras [9, page 106], if G is a

solvable Lie algebra then the algebra $\text{ad}(G^{\mathbb{C}})$ is nilpotent, where $G^{\mathbb{C}} = G + \sqrt{-1}G$. Therefore, the commutator G' of G is nilpotent.

Now assume that G is solvable and of dimension three.

(a) If H is a two-dimensional algebra with basis $\{X, Y\}$, then its commutator is generated by $[X, Y]$. If H is nonabelian, extend $U = [X, Y]$ to a basis $\{U, V\}$ of H . Then, scaling V , we have the canonical representation of H by the relations $[V, U] = U$; such an algebra is not nilpotent.

Therefore if G' is two-dimensional, it must be abelian. Take a basis $\{X, Y\}$ of G' and extend it to a basis $\{X, Y, Z\}$ of G' . Then $\text{ad}(Z)$ operating on G' does not have 0 as an eigenvalue and the eigenvalues and their multiplicities of $\text{ad}(Z)$ operating on G' completely determine the structure of G .

(b) Assume that G' is one-dimensional. Let $\{U\}$ be a basis of G' . Extend it to a basis of $\{X, Y, U\}$ of G .

So $[X, Y] = aU, [X, U] = bU, [Y, U] = cU$. Now $\dim Z_G(U) \geq 1$. Suppose it is one. Then $b, c \neq 0$. By scaling X, Y suitably, we then have $[X, Y] = aU, [X, U] = U, [Y, U] = U$ so $[X - Y, U] = 0$. Therefore $X - Y, U$ are in $Z_G(U)$ and $\dim Z_G(U)$ is at least two.

(i) Suppose $\dim Z_G(U) = 2$. Choose a basis $\{Y, U\}$ of $Z_G(U)$ and extend it to a basis $\{X, Y, U\}$ of G . So $[X, Y] = aU, [X, U] = bU, [Y, U] = 0$, with $b \neq 0$. By scaling, we have that $[X, Y] = aU, [X, U] = U, [Y, U] = 0$.

Now, for any λ we have

$$[X, Y + \lambda U] = aU + \lambda U, [X, U] = U, [Y + \lambda U, U] = 0.$$

Choosing $a + \lambda = 0$ and renaming $Y - aU$ as Y we have the canonical relations $[X, Y] = 0, [X, U] = U, [Y, U] = 0$ and $Z_G(U) = \langle U, Y \rangle$.

(ii) Suppose $\dim Z_G(U) = 3$. Take any basis, $\{X, Y, U\}$ of G . We then have

$$[X, Y] = aU, [X, U] = 0, [Y, U] = 0, \text{ with } a \neq 0.$$

By scaling, we can assume that $[X, Y] = U, [X, U] = 0, [Y, U] = 0$.

Therefore the canonical representations of G when $\dim G'$ is one are:

$$[X, Y] = 0, [X, U] = U, [Y, U] = 0 \text{ (dim } Z_G(G') \text{ is 2)}$$

and

$$[X, Y] = U, [X, U] = 0, [Y, U] = 0 \text{ (dim } Z_G(G') \text{ is 3)}.$$

□

Corollary 2.2. *If G' is two-dimensional then it is abelian and there is a basis $\{X, Y, Z\}$ of G with $\{X, Y\}$ a basis of G' with $\text{ad}(Z)$ operating on G' as follows - recall that $\text{ad}(Z)$ does not have 0 as an eigenvalue of G' :*

(i) $\text{ad}(Z)$ has real and distinct eigenvalues.

If X, Y are eigenvectors, then as $\text{ad}(Z)$ does not have 0 as an eigenvalue of G' , by scaling Z we have $[Z, X] = X, [Z, Y] = cY, c \neq 0, 1$.

(ii) $\text{ad}(Z)$ has only one eigenvalue and the corresponding eigenspace is two-dimensional.

In this case, scaling Z , we have the canonical representation

$$[Z, X] = X, [Z, Y] = Y.$$

(iii) $\text{ad}(Z)$ has only one real eigenvalue and the corresponding eigenspace is one-dimensional. There is a basis $\{X, Y\}$ of G' with $[Z, X] = X + Y, [Z, Y] = Y$.

(iv) $\text{ad}(Z)$ has a complex eigenvalue λ . The canonical relations - after scaling Z - are

$$[Z, X] = \cos \theta X + \sin \theta Y, [Z, Y] = -\sin \theta X + \cos \theta Y,$$

where X, Y are the real and imaginary parts of an eigenvector for the eigenvalue λ .

Proof. Only cases (iii) and (iv) need proofs.

Case (iii): $\text{ad}(Z)$ has only one real eigenvalue, which is non-zero and the null-space of $\text{ad}(Z) - \lambda I$ is one-dimensional. Let V be an eigenvector for the eigenvalue λ . Extend V to a basis $\{X, V\}$ of G' . Then X is a generalized eigenvector of Z and $\{X, Y = \text{ad}(Z - \lambda I)(X)\}$ is a basis of G' .

We have $[Z, X] = \lambda X + Y, [Z, Y] = \lambda Y$. Dividing Z by λ and relabeling it Z we have the relations $[Z, X] = X + cY, (c \neq 0), [Z, Y] = Y$. Finally, replacing Y by cY , we have the canonical relations $[Z, X] = X + Y, [Z, Y] = Y$.

Case (iv): if one of the eigenvalues of $\text{ad}(Z)$ is complex but not real, then the other eigenvalue is its conjugate. Denote by λ any one of these eigenvalues. Then the eigenvectors live in the Lie algebra $G + \sqrt{-1}G$ - the complexification of G . We will denote the complexification of any Lie algebra G by $G^{\mathbb{C}}$. The eigenvectors of $\text{ad}(Z)$ live in $G^{\mathbb{C}}$. Find an eigenvector e for $\text{ad}(Z)$.

The real and imaginary parts of e are $\text{Re}(e) = \frac{e + \bar{e}}{2}, \text{Im}(e) = \frac{e - \bar{e}}{2i}$.

By scaling Z by $\frac{1}{|\lambda|}$, we have the canonical relations

$$[Z, \operatorname{Re}(e)] = \cos \theta \operatorname{Re}(e) + \sin \theta \operatorname{Im}(e)$$

$$[Z, \operatorname{Im}(e)] = -\sin \theta \operatorname{Re}(e) + \cos \theta \operatorname{Im}(e).$$

Take $X = \operatorname{Re}(e)$ and $Y = \operatorname{Im}(e)$. □

3 Local classification of commuting Lie algebras of vector fields in \mathbb{R}^2

The algorithms for bringing a given ODE to its canonical form by point transformations ultimately reduce to constructive classifications of abelian Lie algebras of vector fields. This section is devoted to an algorithmic procedure for finding canonical forms of such algebras.

It is well-known that if X is a vector field and $X(p) \neq 0$, then near p we can introduce coordinates in which $X = \partial_x$. To find such a canonical coordinate, one uses the method of characteristics [7, page 142], to find a basic invariant function y of X and choose a function x functionally independent from y . Then in these coordinates $X = f(x, y) \partial_x$, with f non-vanishing near p .

We want functions \tilde{x}, \tilde{y} with $X(\tilde{x}) = 1, X(\tilde{y}) = 0$. Then the requirements become $f(x, y) \partial_x(\tilde{x}) = 1, f(x, y) \partial_x(\tilde{y}) = 0$. A solution of this system is $\tilde{x} = \int \frac{dx}{f(x, y)}, \tilde{y} = y$.

Now suppose that X, Y are commuting vector fields with rank 2 near a point p ; so that $X(q), Y(q)$ are linearly independent near p ; say $X(p) \neq 0$. We may therefore assume that in some neighbourhood of $p, X(q) \neq 0$ and X, Y of rank 2 in this neighbourhood.

Choose local coordinates x, y in possibly a smaller neighbourhood with $X = \partial_x$. As Y operates on invariants of $X, Y(y) = g(y)$. Now as Y operates non-trivially on invariants of X because of the rank condition, we can find a function \tilde{y} of y with $Y(\tilde{y}) = 1$. By change of notation, we now have coordinates x, y with $X = \partial_x, Y = \xi(y) \partial_x + \partial_y$. We want new coordinates \tilde{x}, \tilde{y} with $X(\tilde{x}) = 1, X(\tilde{y}) = 0, Y(\tilde{x}) = 0, Y(\tilde{y}) = 1$. The system to solve now is $\frac{\partial \tilde{x}}{\partial x} = 1, \frac{\partial \tilde{y}}{\partial x} = 0, \xi(y) \frac{\partial \tilde{x}}{\partial y} + \frac{\partial \tilde{x}}{\partial y} = 0, \frac{\partial \tilde{y}}{\partial y} = 1$. The solution is given by $\tilde{x} = x + \varphi(y), \frac{\partial \tilde{x}}{\partial y} = \varphi'(y) = -\xi(y), \tilde{y} = y$. If the fields are commuting of rank 1, and $X = \partial_x$ in local coordinates, then $Y = f(y)X$, with $f'(y) \neq 0$. Then in the variables $x, \tilde{y} = f(y)$ the canonical form of the fields is $X = \partial_x, Y = \tilde{y} \partial_x$.

4 Realizations of three-dimensional algebras as vector fields in \mathbb{R}^2

We now turn to realizations of the algebras occurring in Theorem 2.1 and its corollary as algebras of vector fields in \mathbb{R}^2 . This was done by Lie for complex Lie algebras that arise as symmetries of second-order ODEs: see [7, page 164] and [4, pages 479-530]. Mahomed and Leach [5] extended this study for real Lie algebras. Similar ideas apply to vector fields in \mathbb{R}^3 .

Outline of the argument: Using Theorem 2.1, the canonical forms for solvable three-dimensional algebras with nontrivial commutator can be easily obtained algorithmically as one has just to put a two-dimensional algebra, depending on its rank, in canonical form. The reason is that if G' is two-dimensional then it is abelian and its rank determines the canonical form of G . If G' is of dimension one and its centralizer has dimension two, then the rank of the centralizer determines the canonical form of G . Finally, if G' is one-dimensional and its centralizer has dimension three, then rank of G must be 2 and choosing a field supported outside of G' gives an abelian two-dimensional algebra of rank 2 which determines the canonical form of G .

If $G' = G$, then picking any nonzero element X of G , the eigenvalues of $\text{ad}(X)$ determine the canonical form of G and of the vector fields also - as detailed below in section 4.3.

Details of the classification: Assume that G is three-dimensional and abelian. Then its rank is at most 2 and it cannot be 2 as the centralizer of $\{\partial_x, \partial_y\}$ is $\langle \partial_x, \partial_y \rangle$. Therefore it is of rank 1. Pick any non-zero element X of G and find canonical coordinates x, y so that $X = \partial_x$ and extend it to a basis $\{X, Y, Z\}$. Since rank of G is 1, necessarily $Y = f(y)\partial_x$, $Z = g(y)\partial_x$. Since f is not a constant, we can take $\tilde{y} = f(y)$. So the basis becomes $\{\partial_x, \tilde{y}\partial_x, g(f^{-1}(\tilde{y}))\partial_x = h(\tilde{y})\partial_x\}$, where $h(\tilde{y})$ is linearly independent of $\{1, \tilde{y}\}$.

For the canonical realizations of the algebras occurring in Theorem 1, one needs to solve equations of the type $[Z, X] = aX + bY$, $[Z, Y] = cX + dY$, where $X = \partial_x$ and $Y = \partial_y$ or $y\partial_x$. In case $Y = \partial_y$, this is straightforward. However, when $Y = y\partial_x$, we have $Z = \xi\partial_x + \eta\partial_y$, where

$$\xi = -ax - bxy + \varphi(y), \quad \eta = c + (d - a)y - by^2.$$

We want a change of variables \tilde{x}, \tilde{y} so that $\partial_x = \partial_{\tilde{x}}$, $y\partial_x = \tilde{y}\partial_{\tilde{x}}$ and

$$Z = (-a\tilde{x} - b\tilde{x}\tilde{y})\partial_{\tilde{x}} + (c + (d - a)\tilde{y} - b\tilde{y}^2)\partial_{\tilde{y}} \quad (4.1)$$

Then necessarily $\tilde{y} = y$, $\tilde{x} = x + \psi(y)$, $\partial_{\tilde{y}} = -\psi'(y)\partial_x + \partial_y$. Substituting these expressions in the equation (4.1) we arrive at equation

$$\varphi(y) = \psi(y)(-a - by) - \psi'(y)(c + (d - a)y - by^2) \quad (4.2)$$

Solving this differential equation for ψ removes the term $\varphi(y)$ in the field Z in the new variables.

This gives the following realizations of the algebras occurring in Theorem 1 as vector fields in the plane:

4.1 $\dim(G') = 1$

I: $\dim Z_G(G') = 2$, $[X, Y] = 0$, $[X, U] = U$, $[Y, U] = 0$, $\text{rank } Z_G(G') = 1$

$U = \partial_y$, $Y = f(x) \partial_y$, where f is not a constant. Making a change of variables $\tilde{x} = f(x)$, relabeling $\tilde{x} = x$, using (4.1), (4.2)- with Z replaced by X - we obtain the representation

$$U = \partial_y, Y = x \partial_y, X = -y \partial_y - x \partial_x$$

II: $\dim Z_G(G') = 2$, $[X, Y] = 0$, $[X, U] = U$, $[Y, U] = 0$, $\text{rank } Z_G(G') = 2$,

$$U = \partial_y, Y = \partial_x, X = -y \partial_y$$

III: $\dim Z_G(G') = 3$, $[X, Y] = U$, $[X, U] = 0$, $[Y, U] = 0$

In this case rank of G must be 2. In the canonical coordinates for U , we have $U = \partial_y$, and one of X or Y is supported outside ∂_y , otherwise $[X, Y]$ would be 0. By symmetry between X and Y , we may suppose that X is supported outside ∂_y and therefore $\text{rank} \langle X, U \rangle = 2$. We may then suppose that $X = \partial_x$. This determines $Y = x \partial_y$. The canonical realization is thus $X = \partial_x, U = \partial_y, Y = x \partial_y$.

4.2 $\dim(G') = 2$

(a) There is a basis $\{X, Y, Z\}$ of G with $\{X, Y\}$ a basis of G' with $\text{ad}(Z)$ operating on G' having real and distinct eigenvalues. Computing the eigenvectors for these eigenvalues and labeling them X, Y and dividing Z by the eigenvalue for the eigenvector X , and relabeling it Z we have the relations $[Z, X] = X$, $[Z, Y] = cY$, $c \neq 0, 1$.

IV: $\text{rank}(G') = 2$.

Choosing coordinates with $X = \partial_x, Y = \partial_y$ we get $Z = -x \partial_x - cy \partial_y$

V: $\text{rank}(G') = 1$.

Choosing coordinates in which $X = \partial_y, Y = x \partial_y$ and using (4.1) and (4.2) we obtain - by change of notation $Z = (c - 1) x \partial_x - y \partial_y$.

(b) There is a basis $\{X, Y, Z\}$ of G with $\{X, Y\}$ a basis of G' with $\text{ad}(Z)$ operating on G' having a real eigenvalue with the corresponding eigenspace of dimension two. Computing the eigenvectors

for these eigenvalues and labeling them X, Y and dividing Z by the eigenvalue and relabeling it Z we have the relations

$$[Z, X] = X, [Z, Y] = Y.$$

VI: rank $(G') = 2$.

Choosing coordinates with $X = \partial_x, Y = \partial_y$ we deduce $Z = -x \partial_x - y \partial_y$

VII: rank $(G') = 1$.

Choosing coordinates with $X = \partial_y, Y = x \partial_y$ and using (4.1) and (4.2) we get - by change of notation - $Z = -y \partial_y$.

(c) There is a basis $\{X, Y, Z\}$ of G with $\{X, Y\}$ a basis of G' with $\text{ad}(Z)$ operating on G' having only one eigenvalue λ with the corresponding eigenspace of dimension one. Find the corresponding eigenvector and let X be a vector linearly independent from this eigenvector. Then X is a generalized eigenvector and $\{X, Y = (\text{ad}(Z) - \lambda I)X\}$ is a basis of G' . Dividing Z by λ and relabeling it Z , we have $[Z, X] = X + \frac{1}{\lambda}Y, [Z, Y] = Y$. Replacing Y by $\frac{1}{\lambda}Y$ in the second equation and finally denoting $\frac{1}{\lambda}Y$ by Y we have the relations $[Z, X] = X + Y, [Z, Y] = Y, [X, Y] = 0$.

VIII: rank $(G') = 2$.

Choosing coordinates with $X = \partial_x, Y = \partial_y$ we arrive at

$$Z = -x \partial_x - (x + y) \partial_y$$

IX: rank $(G') = 1$.

Choosing coordinates in which $X = x \partial_y, Y = \partial_y$ and using (4.1) and (4.2) we find - by change of notation - $Z = \partial_x - y \partial_y$.

(d) There is a basis $\{X, Y, Z\}$ of G with $\{X, Y\}$ a basis of G' and $\text{ad}(Z)$ operating on G' having a non-real complex eigenvalue λ . Let e be an eigenvector. Dividing Z by $\frac{1}{|\lambda|}$, denoting it again by Z we have the canonical relations

$$[Z, \text{Re}(e)] = \cos \theta \text{Re}(e) - \sin \theta \text{Im}(e)$$

$$[Z, \text{Im}(e)] = \sin \theta \text{Re}(e) + \cos \theta \text{Im}(e).$$

Take $X = \text{Re}(e)$ and $Y = \text{Im}(e)$.

X: rank $(G') = 2$.

Choosing coordinates with $X = \partial_x, Y = \partial_y$. Then

$$Z = (-x \cos \theta - y \sin \theta) \partial_x + (x \sin \theta - y \cos \theta) \partial_y.$$

XI: $\text{rank}(G') = 1$.

Choosing coordinates in which $X = \partial_y$, $Y = x \partial_y$ and using (4.1) and (4.2) we obtain - by change of notation - $Z = (1 + x^2) \sin \theta \partial_x + y(x \sin \theta - \cos \theta) \partial_y$

4.3 $\dim(G') = 3$

Let G be a three-dimensional Lie algebra with $G' = G$. If I is any non-zero vector in G , then its centralizer consists of multiples of I ; for if U is linearly independent of I and it centralizes I , then extending I to a basis of G we see that G' is at most two-dimensional. Therefore, the non-zero eigenvalues of $\text{ad}(I)$ occur in pairs $\lambda, -\lambda$ - as the trace of $\text{ad}(I)$ is zero.

Case (i): The non-zero eigenvalues of $\text{ad}(I)$ are real given as $\pm\lambda$. Then we can find eigenvectors U, V of I with $[I, U] = \lambda U$, $[I, V] = -\lambda V$, $[U, V] = cI$ where $c \neq 0$ as $[U, V]$ centralizes I . Setting $X = U$, $Y = \frac{2}{c\lambda} V$, $Z = \frac{2}{\lambda} I$, we have the standard relations of $sl(2, \mathbb{R})$ given by

$$[Z, X] = 2X, [Z, Y] = -2Y, [X, Y] = Z.$$

In this case the Killing form is non-degenerate and indefinite.

Case (ii): $\text{ad}(I)$ has a non-real eigenvalue. In this case, the nonreal eigenvalues must be purely imaginary. Scaling I , we may suppose that the non-zero eigenvalues of $\text{ad}(I)$ are $\pm\sqrt{-1}$. Take an eigenvector e of $\text{ad}(I)$ in the complexification of G with eigenvalue $\sqrt{-1}$. Write $e = U - \sqrt{-1} V$. Then $[I, U] = V$ and $[I, V] = -U$. Now $[e, \bar{e}]$ commutes with I and $[e, \bar{e}] = 2\sqrt{-1}[U, V]$. Therefore $[U, V]$ commutes with I . We can scale these generators such that $[I, U] = V$, $[I, V] = -U$ and $[U, V] = \epsilon I$, where $\epsilon^2 = 1$. If $\epsilon = -1$, then we have the generators with $[I, U] = V$, $[I, V] = -U$ and $[U, V] = -I$. So $[U, V + I] = -(V + I)$ and $[U, V - I] = V - I$ and we are back to case (i). In this case the Killing form is non-degenerate and indefinite. If $\epsilon = 1$, then we have the generators with $[I, U] = V$, $[I, V] = -U$ and $[U, V] = I$. The Killing form is negative definite. These are the standard relations of $so(3)$.

Case (iii): All the eigenvalues of $\text{ad}(I)$ are zero. In this case $\text{ad}(I)$ is nilpotent and in the normalizer $N(I)$ of I there must be an element with real non-zero eigenvalues, specifically, any element H in $N(I)$ complementary to I must have non-zero eigenvalues, so $[H, I] = \lambda I$, $[H, U] = -\lambda U$ for some element U and we are back to case (i).

Remark 4.1. *If the Killing form is definite, then the Lie algebra must be $so(3)$. So its Killing form must in any case be negative definite. In this case the non-zero eigenvalues of $\text{ad}(I)$ must be purely imaginary. Arguing exactly as above, given a non-zero element I of L , we can find generators U, V with $[I, U] = V$, $[I, V] = -U$ and $[U, V] = cI$, where $c = \pm 1$. Now $c = -1$ would*

give an indefinite Killing form, so $[I, U] = V, [I, V] = -U$ and $[U, V] = I$ which give the relations for $so(3)$ for the triple U, V, I . In this case clearly $[I, \sqrt{-1} U] = \sqrt{-1} V, [I, \sqrt{-1} V] = -\sqrt{-1} U$ and $[\sqrt{-1} U, \sqrt{-1} V] = -I$ which are the relations for $sl(2, \mathbb{C})$. On the other hand, if $[I, U] = U, [I, V] = -V, [U, V] = I$, the non-zero eigenvalues of $\text{ad}(U+V)$ are purely imaginary and working with the corresponding eigenvectors we obtain a basis for $so(3, \mathbb{C})$. For this reason, over the complex numbers, there is a correspondence between $sl(2, \mathbb{R})$ and $so(3)$ invariant equations.

4.3.1 Realizations of $sl(2, \mathbb{R})$ as vector fields in \mathbb{R}^2

Here we discuss the indefinite case. So let L be a three-dimensional algebra of vector fields with indefinite Killing form. Then L has a basis $\{X, Y, Z\}$ such that

$$[Z, X] = 2X, [Z, Y] = -2Y, [X, Y] = Z.$$

Find coordinates in which $Z = \partial_x$. Then necessarily X has the following form

$$X = e^{2x} (f_1(y)\partial_x + f_2(y)\partial_y). \quad (4.3)$$

As $\{\partial_x, f_1(y)\partial_x + f_2(y)\partial_y\}$ are commuting vectors, we have the following two cases:

Case A: the rank of $\{\partial_x, f_1(y)\partial_x + f_2(y)\partial_y\}$ is 2. There is a change of variables in which $\partial_x = \partial_{\tilde{x}}$ and $f_1(y)\partial_x + f_2(y)\partial_y = \partial_{\tilde{y}}$. Using section 2, such a change of variables can be given explicitly as follows:

$$\tilde{x} = x - \int \frac{f_1}{f_2} dy, \quad \tilde{y} = \int \frac{1}{f_2} dy. \quad (4.4)$$

Therefore in these coordinates

$$X = e^{2(\tilde{x}+f(\tilde{y}))} \partial_{\tilde{y}}, \quad (4.5)$$

where $f(\tilde{y}) = \int \frac{f_1}{f_2} dy$. In the coordinates $\tilde{\tilde{x}}, \tilde{\tilde{y}}$ in which $\tilde{\tilde{x}} = \tilde{x}$ and $e^{2f(\tilde{y})} \partial_{\tilde{y}} = \partial_{\tilde{\tilde{y}}}$ given by

$$\tilde{\tilde{x}} = \tilde{x}, \quad \tilde{\tilde{y}} = \int e^{-2f(\tilde{y})} d\tilde{y}, \quad (4.6)$$

we have

$$Z = \partial_{\tilde{\tilde{x}}}, \quad X = e^{2\tilde{\tilde{x}}} \partial_{\tilde{\tilde{y}}}, \quad Y = e^{-2\tilde{\tilde{x}}} \left((\tilde{\tilde{y}} + c_1) \partial_{\tilde{\tilde{x}}} + \left((\tilde{\tilde{y}} + c_1)^2 + \epsilon \lambda^2 \right) \partial_{\tilde{\tilde{y}}} \right), \quad (4.7)$$

where $\epsilon \in \{0, 1, -1\}$. So, the new coordinates $\bar{x} = \tilde{\tilde{x}}, \bar{y} = \frac{1}{\lambda}(\tilde{\tilde{y}} + c_1)$ transform X, Y and Z to

$$Z = \partial_{\bar{x}}, \quad X = \frac{1}{\lambda} e^{2\bar{x}} \partial_{\bar{y}}, \quad Y = \lambda e^{-2\bar{x}} (\bar{y} \partial_{\bar{x}} + (\bar{y}^2 + \epsilon) \partial_{\bar{y}}), \quad (4.8)$$

where $\epsilon \in \{0, 1, -1\}$. Finally, the transformation $\hat{x} = e^{-2\bar{x}}, \hat{y} = \bar{y} e^{-2\bar{x}}$ transforms X, Y and Z to the polynomial form

$$X = \partial_{\hat{y}}, Y = -2\hat{x}\hat{y}\partial_{\hat{x}} + (-\hat{y}^2 + \epsilon \hat{x}^2)\partial_{\hat{y}}, Z = -2\hat{x}\partial_{\hat{x}} - 2\hat{y}\partial_{\hat{y}},$$

where $\epsilon \in \{0, 1, -1\}$.

Case B: the rank of $\{\partial_x, f_1(y)\partial_x + f_2(y)\partial_y\}$ is 1. In this case the vector X should have the form

$$X = e^{2x} f_1(y) \partial_x. \quad (4.9)$$

If $f_1(y)$ is constant, then

$$Z = \partial_x, \quad X = e^{2x} \partial_x. \quad (4.10)$$

Using $[X, Y] = Z$, gives

$$Y = -\frac{1}{4} e^{-2x} \partial_x. \quad (4.11)$$

If $f_1(y)$ is not a constant, we can introduce a change of variables

$$\tilde{x} = x, \quad \tilde{y} = f_1(y). \quad (4.12)$$

Therefore in these coordinates

$$X = e^{2(\tilde{x}+f(\tilde{y}))} \partial_{\tilde{x}}, \quad (4.13)$$

where $f(\tilde{y}) = \frac{1}{2} \ln \tilde{y}$. Finally, in the coordinates $\tilde{\tilde{x}}, \tilde{\tilde{y}}$ given by

$$\tilde{\tilde{x}} = \tilde{x} + \frac{1}{2} \ln \tilde{y}, \quad \tilde{\tilde{y}} = \tilde{y}, \quad (4.14)$$

we have

$$Z = \partial_{\tilde{\tilde{x}}}, \quad X = e^{2\tilde{\tilde{x}}} \partial_{\tilde{\tilde{x}}}. \quad (4.15)$$

Using $[X, Y] = Z$, gives

$$Y = -\frac{1}{4} e^{-2\tilde{\tilde{x}}} \partial_{\tilde{\tilde{x}}}. \quad (4.16)$$

Finally, the transformation $\hat{x} = \tilde{\tilde{x}}, \hat{y} = -\frac{1}{2} e^{-2\tilde{\tilde{x}}}$ transforms X, Y and Z to the polynomial form

$$X = \partial_{\hat{y}}, Y = -\hat{y}^2 \partial_{\hat{y}}, Z = -2\hat{y} \partial_{\hat{y}}.$$

4.3.2 Realizations of $so(3)$ as vector fields in \mathbb{R}^2

Let L be a three-dimensional algebra of vector fields with negative definite Killing form. Then L has a basis $\{X, Y, Z\}$ such that

$$[X, Y] = Z, [Y, Z] = X, [Z, X] = Y.$$

Find coordinates in which $X = \partial_x$. Then necessarily $Y - \sqrt{-1} Z$ has the following form

$$Y - \sqrt{-1} Z = e^{\sqrt{-1}x} [(f_1(y)\partial_x + f_2(y)\partial_y) + \sqrt{-1}(f_3(y)\partial_x + f_4(y)\partial_y)]. \quad (4.17)$$

Since the rank of $\{\partial_x, f_1(y)\partial_x + f_2(y)\partial_y, f_3(y)\partial_x + f_4(y)\partial_y\}$ cannot be 1, we can assume without loss of generality that rank of $\{\partial_x, f_1(y)\partial_x + f_2(y)\partial_y\}$ is 2. We may therefore assume that $f_2(y) \neq 0$. As $\partial_x, f_1(y)\partial_x + f_2(y)\partial_y$ are commuting vectors, there is a change of variables in which $\partial_x = \partial_{\tilde{x}}$ and $f_1(y)\partial_x + f_2(y)\partial_y = \partial_{\tilde{y}}$. Using section 2, such a change of variables can be given explicitly as follows:

$$\tilde{x} = x - \int \frac{f_1}{f_2} dy, \quad \tilde{y} = \int \frac{1}{f_2} dy, \quad (4.18)$$

Therefore in these coordinates

$$Y - \sqrt{-1} Z = e^{\sqrt{-1}(\tilde{x}+f(\tilde{y}))} [\partial_{\tilde{y}} + \sqrt{-1} (A(\tilde{y})\partial_{\tilde{x}} + B(\tilde{y})\partial_{\tilde{y}})], \quad (4.19)$$

where $f(\tilde{y}) = \int \frac{f_1}{f_2} dy$, $A(\tilde{y}) = \frac{f_2 f_3 - f_1 f_4}{f_2}$, and $B(\tilde{y}) = \frac{f_4}{f_2}$. Using the fact that $[Y, Z] = \partial_{\tilde{x}}$ if and only if $[Y - \sqrt{-1} Z, Y + \sqrt{-1} Z] = 2\sqrt{-1}\partial_{\tilde{x}}$, the necessary and sufficient conditions for $f(\tilde{y})$, $A(\tilde{y})$ and $B(\tilde{y})$ to give a representation of $so(3)$ are

$$\begin{aligned} A^2 + ABf' + A' &= -1, \\ (AB + B') + (1 + B^2)f' &= 0. \end{aligned} \quad (4.20)$$

To reduce the form (4.19) to the simplest form, we look at the classical Bianchi representation of vector fields on $\mathbb{P}^2(\mathbb{R})$ induced by the rotations on \mathbb{R}^3 . It is given by

$$L_{3;9} : X = \partial_{\tilde{x}}, Y = \tilde{y} \sin \tilde{x} \partial_{\tilde{x}} + (1 + \tilde{y}^2) \cos \tilde{x} \partial_{\tilde{y}}, Z = \tilde{y} \cos \tilde{x} \partial_{\tilde{x}} - (1 + \tilde{y}^2) \sin \tilde{x} \partial_{\tilde{y}}.$$

So

$$Y - \sqrt{-1} Z = e^{\sqrt{-1}\tilde{x}} [(1 + \tilde{y}^2)\partial_{\tilde{y}} - \sqrt{-1}\tilde{y}\partial_{\tilde{x}}]. \quad (4.21)$$

The conditions that the form (4.19) can be written in the form (4.21) with $\partial_{\tilde{x}} = \partial_{\tilde{x}}$ and $\tilde{y} = \psi(\tilde{y})$ are exactly the equations (4.20). This gives the transformation

$$\tilde{\tilde{x}} = \tilde{x} + f + \tan^{-1} B, \quad \tilde{\tilde{y}} = -\frac{A}{\sqrt{1+B^2}}. \quad (4.22)$$

So the transformation

$$\tilde{\tilde{x}} = x + \tan^{-1} \left(\frac{f_4}{f_2} \right), \quad \tilde{\tilde{y}} = \frac{f_1 f_4 - f_2 f_3}{\sqrt{f_2^2 + f_4^2}}. \quad (4.23)$$

maps the form (4.17) to the form (4.21).

In case $f_2(y) = 0$, $f_4(y) \neq 0$, the formula becomes

$$\tilde{\tilde{x}} = x - \tan^{-1} \left(\frac{f_2}{f_4} \right) + \frac{\pi}{2}, \quad \tilde{\tilde{y}} = \frac{f_1 f_4 - f_2 f_3}{\sqrt{f_2^2 + f_4^2}}. \quad (4.24)$$

Hence, up to change of coordinates, there is only one realization as vector fields in \mathbb{R}^2 .

5 Summary of the results

Based on the discussion in the previous section and using the notations in ref. [5], we can state the following theorem:

Theorem 5.1. *Every three-dimensional Lie algebra has one of the following 17 realizations in \mathbb{R}^2 :*

A) $\dim G' = 0$:

Then G is abelian of rank 1 and there are infinitely many realizations:

$L_{3;1} : X = \partial_y, Y = x\partial_y, Z = f(x)\partial_y$ where $f(x)$ is linearly independent of $\{1, x\}$.

B) $\dim (G') = 2$:

Then the eigenvalues of G/G' on G' never zero and there are the following eight cases:

1) $\text{rank}(G') = 1$ and the eigenvalues of G/G' on G' are real and distinct.

$L_{3;6}^{II} : X = \partial_y, Y = x\partial_y, Z = (c-1)x\partial_x - y\partial_y, c \neq 0, 1$

with the nonzero commutators $[Z, X] = X, [Z, Y] = cY, c \neq 0, 1$.

2) $\text{rank}(G') = 2$ and the eigenvalues of G/G' on G' are real and distinct.

$L_{3;6}^I : X = \partial_x, Y = \partial_y, Z = -x\partial_x - cy\partial_y, c \neq 0, 1$

with the nonzero commutators $[Z, X] = X, [Z, Y] = cY, c \neq 0, 1$.

3) $\text{rank}(G') = 1$ and the eigenvalues of G/G' on G' are real and repeated with eigenspace of dimension 2.

$L_{3;5}^{II} : X = \partial_y, Y = x\partial_y, Z = -y\partial_y$

with the nonzero commutators $[Z, X] = X, [Z, Y] = Y$.

4) $\text{rank}(G') = 2$ and the eigenvalues of G/G' on G' are real and repeated with eigenspace of dimension 2.

$L_{3;5}^I : X = \partial_x, Y = \partial_y, Z = -x\partial_x - y\partial_y$

with the nonzero commutators $[Z, X] = X, [Z, Y] = Y$.

5) $\text{rank}(G') = 1$ and the eigenvalues of G/G' on G' are real and repeated with eigenspace of dimension 1.

$L_{3;3}^{II} : X = x\partial_y, Y = \partial_y, Z = \partial_x - y\partial_y$

with the nonzero commutators $[Z, X] = X + Y, [Z, Y] = Y$

6) $\text{rank}(G') = 2$ and the eigenvalues of G/G' on G' are real and repeated with eigenspace of dimension 1.

$L_{3;3}^I : X = \partial_x, Y = \partial_y, Z = -x\partial_x - (x+y)\partial_y$

with the nonzero commutators $[Z, X] = X + Y, [Z, Y] = Y$

7) $\text{rank}(G') = 1$ and the eigenvalues of G/G' on G' are complex.

$L_{3;7}^{II} : X = \partial_y, Y = x\partial_y, Z = \sin\theta(1+x^2)\partial_x + y(x\sin\theta - \cos\theta)\partial_y$

with the nonzero commutators $[Z, X] = \cos\theta X - \sin\theta Y, [Z, Y] = \sin\theta X + \cos\theta Y$.

8) $\text{rank}(G') = 2$ and the eigenvalues of G/G' on G' are complex.

$L_{3:7}^I : X = \partial_x, Y = \partial_y, Z = (-x \cos \theta - y \sin \theta) \partial_x + (x \sin \theta - y \cos \theta) \partial_y$
with the nonzero commutators $[Z, X] = \cos \theta X - \sin \theta Y, [Z, Y] = \sin \theta X + \cos \theta Y$.

C) $\dim(G') = 1$

1) The centralizer $Z_G(G')$ is 2 dimensional, then there are the two cases:

(i) $\text{rank}(Z_G(G'))$ is 1.

$L_{3:4}^{II} : X = -x\partial_x - y\partial_y, Y = x\partial_y, Z = \partial_y$
with the nonzero commutator $[X, Z] = Z$.

(ii) $\text{rank}(Z_G(G'))$ is 2.

$L_{3:4}^I : X = -y\partial_y, Y = \partial_x, Z = \partial_y$
with the nonzero commutator $[X, Z] = Z$.

2) The centralizer $Z_G(G')$ is three-dimensional:

$L_{3:2} : X = \partial_x, Y = x\partial_y, Z = \partial_y$
with the nonzero commutator $[X, Y] = Z$.

D) $G' = G$:

1) The Killing form is negative definite.

The Lie algebra is $so(3)$ and there is one realization:

$L_{3:9} : X = \partial_x, Y = y \sin x \partial_x + (1 + y^2) \cos x \partial_y, Z = y \cos x \partial_x - (1 + y^2) \sin x \partial_y$
with the nonzero commutators $[X, Y] = Z, [Y, Z] = X, [Z, X] = Y$.

2) The Killing form is indefinite.

The Lie algebra is $sl(2, R)$ and there are two cases:

(i) The rank of the generators is 2 and there is one of the following three realizations.

$L_{3:8}^I : X = \partial_y, Y = -2xy\partial_x - y^2\partial_y, Z = -2x\partial_x - 2y\partial_y,$
 $L_{3:8}^{II} : X = \partial_y, Y = -2xy\partial_x + (-y^2 + x^2)\partial_y, Z = -2x\partial_x - 2y\partial_y,$
 $L_{3:8}^{III} : X = \partial_y, Y = -2xy\partial_x - (y^2 + x^2)\partial_y, Z = -2x\partial_x - 2y\partial_y,$
with the nonzero commutators $[Z, X] = 2X, [Z, Y] = -2Y, [X, Y] = Z$.

(ii) The rank of the generators is 1 and there is one realization.

$L_{3:8}^{IV} : X = \partial_y, Y = -y^2\partial_y, Z = -2y\partial_y,$
with the nonzero commutators $[Z, X] = 2X, [Z, Y] = -2Y, [X, Y] = Z$.

6 Illustrative examples on the 17 Bianchi types

The equations considered here have been obtained by determining joint invariants of appropriate order for all the Lie-Bianchi types and transforming the equations by simple point transformations. The point of these examples is to recover something close to the inverse of these transformations

algorithmically.

By computing the joint invariants of the realizations of Bianchi types, we see that there are no second-order invariant ODEs when $\text{rank } G' = 1$. However, there are higher-order invariant ODEs for such cases. Even when $\text{rank } G' = 2$, not all of Bianchi types have second-order invariant ODEs. For this reason, we give a procedure illustrated by examples given below for each of the Bianchi types which works in principle for any ODE of arbitrary order that admits a three-dimensional symmetry algebra to reduce it to its canonical form.

Example 6.1. $L_{3:1}$

Consider the ODE

$$v^{(4)} = \frac{1}{v'^5} \left(-vv'^{10} + 10v'^4v''v''' - 15v'^3v''^3 - v'^2v''^2 + 6v'v''^2v''' - 9v''^4 \right) \quad (6.25)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial u}, \quad Y_2 = v \frac{\partial}{\partial u}, \quad Y_3 = v^2 \frac{\partial}{\partial u}, \quad (6.26)$$

Since G is abelian of rank 1, using Theorem 5.1, the fourth-order ODE (6.25) can be transformed to the canonical form of $L_{3:1}$ via a point transformation.

In order to construct such a point transformation, one needs to match the the symmetries with the realizations of the Lie algebra of $L_{3:1}$ given by Theorem 5.1 in the following way:

$$X = Y_1, \quad Y = Y_2, \quad Z = Y_3. \quad (6.27)$$

Applying this correspondence to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_y = 1, \quad x\phi_y = \psi, \quad f(x)\phi_y = \psi^2, \quad \psi_y = 0, \quad x\psi_y = 0, \quad f(x)\psi_y = 0. \quad (6.28)$$

The solution of the system (6.28) gives the following point transformation

$$u = y, \quad v = x, \quad (6.29)$$

for $f(x) = x^2$ which transforms ODE (6.25) to its canonical form

$$y^{(4)} = \left(\frac{f^{(4)}}{f''} \right) y'' + g \left(x, y''' - \left(\frac{f'''}{f''} \right) y'' \right), \quad (6.30)$$

with $g(z, w) = z + w^2$.

Example 6.2. $L_{3:2}$

Consider the ODE

$$v''' = - (v' - v'')^3 e^{-3u} + e^{3u} - 2v' + 3v'', \quad (6.31)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial v}, \quad Y_2 = e^u \frac{\partial}{\partial v}, \quad Y_3 = e^{-u} \frac{\partial}{\partial u}, \quad (6.32)$$

with the nonzero commutators

$$[Y_2, Y_3] = -Y_1. \quad (6.33)$$

Since $\dim G' = 1$, $\dim Z_G(G') = 3$, using Theorem 5.1, the third-order ODE (6.31) can be transformed to the canonical form of $L_{3:2}$ via a point transformation.

In order to construct such a point transformation, one needs to match any vector from $G' = \langle Y_1 \rangle$ with $Z = \frac{\partial}{\partial y}$ and any vector which is functionally independent of G' with $X = \frac{\partial}{\partial x}$ in the following way:

$$Z = Y_1, \quad X = Y_3. \quad (6.34)$$

Applying this correspondence to the point transformation $u = \phi(x, y)$, $v = \psi(x, y)$ yields the system

$$\phi_x = e^{-\phi}, \quad \phi_y = 0, \quad \psi_x = 0, \quad \psi_y = 1. \quad (6.35)$$

The solution of the system (6.35) gives the following point transformation

$$u = \ln x, \quad v = y, \quad (6.36)$$

which transforms ODE (6.31) to its canonical form

$$y''' = f(y''), \quad (6.37)$$

with $f(z) = z^3 + 1$.

Example 6.3. $L_{3:3}^I$

Consider the ODE

$$v'' = -\frac{1}{3}(v' - 2)^3 \exp\left(\frac{v' + 1}{v' - 2}\right) \quad (6.38)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial v}, \quad Y_2 = \frac{\partial}{\partial u}, \quad Y_3 = (5u - v) \frac{\partial}{\partial u} + (4u + v) \frac{\partial}{\partial v}, \quad (6.39)$$

with the nonzero commutators

$$[Y_1, Y_3] = Y_1 - Y_2, \quad [Y_2, Y_3] = 4Y_1 + 5Y_2. \quad (6.40)$$

Here $\dim G' = 2$, $\text{rank}(G') = 2$ and the adjoint action of $G/G' = \langle \bar{Y}_3 \rangle$ on $G' = \langle Y_1, Y_2 \rangle$ given by

$$\text{ad}(\bar{Y}_3) = \begin{pmatrix} -1 & -4 \\ 1 & -5 \end{pmatrix} \quad (6.41)$$

has $\lambda = -3$ as a repeated real eigenvalue. The vector $2Y_1 + Y_2$ is an eigenvector and Y_1 is a generalized eigenvector because, in two dimensions, any vector linearly independent of the eigenvector is a generalized eigenvector. Using Theorem 5.1, the second-order ODE (6.38) can be transformed to the canonical form of $L_{3:3}^I$ via a point transformation.

In order to construct such a point transformation, one needs to match the the generalized eigenvector with $X = \frac{\partial}{\partial x}$ and the scaled eigenvector by $\frac{1}{\lambda}$ with $Y = \frac{\partial}{\partial y}$, in the following way:

$$X = Y_1, \quad Y = -\frac{1}{3}(2Y_1 + Y_2). \quad (6.42)$$

Applying this correspondence to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_x = 0, \quad \phi_y = -\frac{1}{3}, \quad \psi_x = 1, \quad \psi_y = -\frac{2}{3}. \quad (6.43)$$

The solution of the system (6.43) gives the following point transformation

$$u = -\frac{1}{3}y, \quad v = x - \frac{2}{3}y, \quad (6.44)$$

which transforms ODE (6.38) to its canonical form

$$y'' = C \exp(-y') \quad (6.45)$$

with $C = -e$.

Example 6.4. $L_{3:3}^{II}$

Consider the ODE

$$v''' = \frac{1}{u} (e^{-u} \ln((uv'' + 2v')e^u - u) - e^{-u}u - 3v'') \quad (6.46)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial v}, \quad Y_2 = \frac{1}{u} \frac{\partial}{\partial v}, \quad Y_3 = \frac{\partial}{\partial u} + \left(\frac{e^{-u}-v-uv}{u} \right) \frac{\partial}{\partial v}, \quad (6.47)$$

with the nonzero commutators

$$[Y_1, Y_3] = -Y_2 - Y_1, \quad [Y_2, Y_3] = -Y_2. \quad (6.48)$$

Here $\dim G' = 2$, $\text{rank}(G') = 1$ and the adjoint action of $G/G' = \langle \bar{Y}_3 \rangle$ on $G' = \langle Y_1, Y_2 \rangle$ given by

$$\text{ad}(\bar{Y}_3) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad (6.49)$$

has $\lambda = 1$ as a repeated real eigenvalue. Here Y_2 is an eigenvector and Y_1 is a generalized eigenvector for the same reason as explained in example 6.3. Using the Theorem 5.1, the second-order ODE (6.46) can be transformed to the canonical form of $L_{3:3}^{II}$ via a point transformation.

In order to construct such a point transformation, one needs to match the the generalized eigenvector with $X = x \frac{\partial}{\partial y}$ and the eigenvector scaled by $\frac{1}{\lambda}$ with $Y = \frac{\partial}{\partial y}$ in the following way:

$$X = Y_1, \quad Y = Y_2. \quad (6.50)$$

Applying this correspondence to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_y = 0, \quad x\phi_y = 0, \quad \psi_y = \frac{1}{\phi}, \quad x\psi_y = 1. \quad (6.51)$$

The solution of the system (6.51) gives the following point transformation

$$u = x, \quad v = \frac{y}{x}, \quad (6.52)$$

which transforms the vector Y_3 which is linearly independent of G' to

$$Z = \frac{\partial}{\partial x} + (-y + f(x)) \frac{\partial}{\partial y}, \quad (6.53)$$

with $f(x) = e^{-x}$. Such a function $f(x)$ can be absorbed using the transformation

$$\tilde{x} = x, \quad \tilde{y} = y - e^{-x} \int e^x f(x) dx = y - xe^{-x}. \quad (6.54)$$

Finally, the composition of the transformations (6.52) and (6.54) transforms ODE (6.46) to its canonical form

$$\tilde{y}''' = e^{-\tilde{x}} g(e^{\tilde{x}} \tilde{y}''), \quad (6.55)$$

with $g(z) = -3 + \ln(z - 2)$.

Example 6.5. $L_{3:4}^I$

Consider the ODE

$$v''' = -\frac{v'^4 - 3v''^2}{v'} - \frac{e^{-v} (v'^2 - v'')^3}{v'^2 (uv' + 1)^3} \quad (6.56)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = -u \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad Y_2 = e^{-v} \frac{\partial}{\partial u}, \quad Y_3 = -ue^{-v} \frac{\partial}{\partial u} + e^{-v} \frac{\partial}{\partial v}, \quad (6.57)$$

with the nonzero commutators

$$[Y_1, Y_3] = -Y_3. \quad (6.58)$$

Since $\dim G' = 1$, $\dim Z_G(G') = 2$ and $\text{rank}(Z_G(G')) = 2$, using Theorem 5.1, the third-order ODE (6.56) can be transformed to the canonical form of $L_{3:4}^I$ via a point transformation.

In order to construct such a point transformation, one needs to match any vector from $G' = \langle Y_3 \rangle$ with $Z = \frac{\partial}{\partial y}$ and any vector from $Z_G(G') = \langle Y_3, Y_2 \rangle$ which is linearly independent of G' with $Y = \frac{\partial}{\partial x}$ in the following way:

$$Z = Y_3, \quad Y = Y_2. \quad (6.59)$$

Applying this correspondence to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_x = e^{-\psi}, \quad \phi_y = -\phi e^{-\psi}, \quad \psi_x = 0, \quad \psi_y = e^{-\psi}. \quad (6.60)$$

The solution of the system (6.60) gives the following point transformation

$$u = \frac{x}{y}, \quad v = \ln y, \quad (6.61)$$

which transforms ODE (6.56) to its canonical form

$$y''' = y' f\left(\frac{y''}{y'}\right), \quad (6.62)$$

with $f(z) = z^3 + 3z^2$.

Example 6.6. $L_{3:4}^{II}$

Consider the ODE

$$v''' = \frac{(32v^6 - 1)v'^8 - 8v^4v'^5v'' + 12v^5v'^3v''^2 - 3v^3v''^3}{4v^5v'^4 - v^3v'v''} \quad (6.63)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial u}, \quad Y_2 = v \frac{\partial}{\partial u}, \quad Y_3 = (u - v^4) \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \quad (6.64)$$

with the nonzero commutators

$$[Y_1, Y_3] = Y_1. \quad (6.65)$$

Since $\dim G' = 1$, $\dim Z_G(G') = 2$ and $\text{rank}(Z_G(G')) = 1$, using Theorem 5.1, the third-order ODE (6.63) can be transformed to the canonical form of $L_{3:4}^{II}$ via a point transformation.

In order to construct such a point transformation, one needs to match any vector from $G' = \langle Y_1 \rangle$ with $Z = \frac{\partial}{\partial y}$ and any vector from $Z_G(G') = \langle Y_1, Y_2 \rangle$ which is linearly independent of G' with $Y = x \frac{\partial}{\partial y}$, in the following way:

$$Z = Y_1, \quad Y = Y_2. \quad (6.66)$$

Applying this correspondence to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_y = 1, \quad x\phi_y = \psi, \quad \psi_y = 0, \quad x\psi_y = 0. \quad (6.67)$$

The solution of the system (6.67) gives the following point transformation

$$u = y, \quad v = x, \quad (6.68)$$

which transforms the vector $-Y_3$ which is linearly independent of $Z_G(G')$ to

$$X = -x \frac{\partial}{\partial x} + (-y + f(x)) \frac{\partial}{\partial y}, \quad (6.69)$$

with $f(x) = x^4$. Such a function $f(x)$ can be absorbed using the transformation

$$\tilde{x} = x, \quad \tilde{y} = y + x \int \frac{f(x)}{x^2} dx = y + \frac{1}{3}x^4. \quad (6.70)$$

Finally, the composition of the transformations (6.68) and (6.70) transforms ODE (6.63) to its canonical form

$$\tilde{y}''' = \frac{1}{\tilde{x}^2} g(\tilde{x}\tilde{y}''), \quad (6.71)$$

with $g(z) = \frac{1}{z}$.

Example 6.7. $L_{3:5}^I$

Consider the ODE

$$v''' = \frac{1}{v'^4} \left(- \left(v''^2 + v'' \right)^2 e^{-3v} + 2v'^7 + 3v''v'^5 + 3v'^3v''^2 \right) \quad (6.72)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial u}, \quad Y_2 = u \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad Y_3 = e^{-v} \frac{\partial}{\partial v}, \quad (6.73)$$

with the nonzero commutators

$$[Y_1, Y_2] = Y_1, \quad [Y_2, Y_3] = -Y_3. \quad (6.74)$$

Here $\dim G' = 2$, $\text{rank}(G') = 2$ and the adjoint action of $G/G' = \langle \bar{Y}_2 \rangle$ on $G' = \langle Y_1, Y_3 \rangle$ is given by

$$\text{ad}(\bar{Y}_2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.75)$$

So $\lambda = -1$ is a repeated real eigenvalue with eigenspace of dimension 2. Using Theorem 5.1, the third-order ODE (6.72) can be transformed to the canonical form of $L_{3:5}^I$ via a point transformation.

In order to construct such a point transformation, one needs to match any two linearly independent vectors of G' with $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$. For example, one can try the obvious choices:

$$X = Y_1, \quad Y = Y_3, \quad (6.76)$$

or the opposite

$$X = Y_3, \quad Y = Y_1. \quad (6.77)$$

Applying the correspondence (6.76) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_x = 1, \quad \phi_y = 0, \quad \psi_x = 0, \quad \psi_y = e^{-\psi}. \quad (6.78)$$

The solution of the system (6.78) gives the following point transformation

$$u = x, \quad v = \ln y, \quad (6.79)$$

which transforms ODE (6.72) to its canonical form

$$y''' = f(y')y''^2 \quad (6.80)$$

with $f(z) = \frac{3z^3-1}{z^4}$.

Similarly, applying the correspondence (6.77) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_x = 0, \quad \phi_y = 1, \quad \psi_x = e^{-\psi}, \quad \psi_y = 0. \quad (6.81)$$

The solution of the system (6.81) gives the following point transformation

$$u = y, \quad v = \ln x, \quad (6.82)$$

which transforms ODE (6.72) to its canonical form

$$y''' = f(y')y''^2 \quad (6.83)$$

with $f(z) = z^2$.

Example 6.8. $L_{3:5}^{II}$

Consider the ODE

$$v^{(4)} = \frac{4vv'^{12} - 4vv'^9v'' + vv'^6v''^2 + 10v'^2v''v'''^2 - 45v'v''^3v''' + 45v''^5}{v'^2(v'''v' - 3v''^2)} \quad (6.84)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial u}, \quad Y_2 = v \frac{\partial}{\partial u}, \quad Y_3 = (u + v^2) \frac{\partial}{\partial u}, \quad (6.85)$$

with the nonzero commutators

$$[Y_1, Y_3] = Y_1, \quad [Y_2, Y_3] = Y_2. \quad (6.86)$$

Here $\dim G' = 2$, $\text{rank}(G') = 1$ and the adjoint action of $G/G' = \langle \bar{Y}_3 \rangle$ on $G' = \langle Y_1, Y_2 \rangle$ is given by

$$\text{ad}(\bar{Y}_3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.87)$$

We have $\lambda = -1$ as a repeated real eigenvalue with eigenspace of dimension 2. Using Theorem 5.1, the fourth-order ODE (6.84) can be transformed to the canonical form of $L_{3:5}^{II}$ via a point transformation.

In order to construct such a point transformation, one needs to match any two linearly independent vectors of G' with $X = \frac{\partial}{\partial y}$ and $Y = x \frac{\partial}{\partial y}$. For example, one can try the obvious choice:

$$X = Y_1, \quad Y = Y_2, \quad (6.88)$$

Applying the correspondence (6.88) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_y = 1, \quad x\phi_y = \psi, \quad \psi_y = 0, \quad x\psi_y = 0. \quad (6.89)$$

The solution of the system (6.89) gives the following point transformation

$$u = y, \quad v = x, \quad (6.90)$$

which transforms the vector $-Y_3$ which is linearly independent of G' to

$$Z = (-y + f(x)) \frac{\partial}{\partial y}, \quad (6.91)$$

with $f(x) = -x^2$. The function $f(x)$ can be absorbed using the transformation

$$\tilde{x} = x, \quad \tilde{y} = y - f(x) = y + x^2. \quad (6.92)$$

Finally, the composition of the transformations (6.90) and (6.92) transforms ODE (6.84) to its canonical form

$$\tilde{y}^{(4)} = \tilde{y}'' g\left(x, \frac{\tilde{y}'''}{\tilde{y}''}\right), \quad (6.93)$$

with $g(z, w) = \frac{z}{w}$.

Example 6.9. $L_{3.6}^I$

Consider the ODE

$$v'' = \frac{1}{9}v^{-4} \left(18v^3v'^2 - \sqrt{v^2 - 2v'}(v^2 + v')^{5/2} \right) \quad (6.94)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial u}, \quad Y_2 = v^2 \frac{\partial}{\partial v}, \quad Y_3 = \left(u + \frac{4}{5}v\right) \frac{\partial}{\partial u} - \left(\frac{2}{5}uv^2 + \frac{7}{5}v\right) \frac{\partial}{\partial v}, \quad (6.95)$$

with the nonzero commutators

$$[Y_1, Y_3] = Y_1 - \frac{2}{5}Y_2, \quad [Y_2, Y_3] = -\frac{4}{5}Y_1 + \frac{7}{5}Y_2. \quad (6.96)$$

We have $\dim G' = 2$, $\text{rank}(G') = 2$ and the adjoint action of $G/G' = \langle \bar{Y}_3 \rangle$ on $G' = \langle Y_1, Y_2 \rangle$ given by

$$\text{ad}(\bar{Y}_3) = \begin{pmatrix} -1 & \frac{4}{5} \\ \frac{2}{5} & -\frac{7}{5} \end{pmatrix} \quad (6.97)$$

has $\lambda_1 = -\frac{3}{5}$ and $\lambda_2 = -\frac{9}{5}$ as distinct real eigenvalues. The corresponding eigenvectors are $2Y_1 + Y_2$ and $Y_2 - Y_1$ respectively. Using the Theorem 5.1, the second-order ODE (6.94) can be transformed to the canonical form of $L_{3,6}^I$ via a point transformation.

In order to construct such a point transformation, one needs to match the the two eigenvectors of $\text{ad}(G/G')$ on G' with constant multiples $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ respectively, in the following way:

$$X = r(2Y_1 + Y_2), \quad Y = s(Y_2 - Y_1), \quad r, s \in \mathbb{R} \setminus \{0\}. \quad (6.98)$$

Applying the correspondence (6.98) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_x = 2r, \quad \phi_y = -s, \quad \psi_x = r\psi^2, \quad \psi_y = s\psi^2. \quad (6.99)$$

The solution of the system (6.99) gives the following point transformation

$$u = 2rx - sy, \quad v = \frac{-1}{rx+sy}, \quad (6.100)$$

which transforms ODE (6.94) to its canonical form

$$y'' = Cy'^{\frac{c-2}{c-1}}, \quad c \neq 0, \frac{1}{2}, 1, 2, \quad (6.101)$$

with $C = r^{\frac{3}{2}}(-s)^{-\frac{1}{2}}$ and $c = \frac{\lambda_2}{\lambda_1} = 3$.

Example 6.10. $L_{3,6}^{II}$

Consider the ODE

$$v''' = \frac{1}{2}(-4v' + 6v'' - 8) + \frac{1}{2}(v' - v'' + 2)^2 e^{2u} \quad (6.102)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial v}, \quad Y_2 = e^u \frac{\partial}{\partial v}, \quad Y_3 = \frac{\partial}{\partial u} - (4u + 2v) \frac{\partial}{\partial v}, \quad (6.103)$$

with the nonzero commutators

$$[Y_1, Y_3] = -2Y_1, \quad [Y_2, Y_3] = -3Y_2. \quad (6.104)$$

Here $\dim G' = 2$, $\text{rank}(G') = 1$ and the adjoint action of $G/G' = \langle \bar{Y}_3 \rangle$ on $G' = \langle Y_1, Y_2 \rangle$ has $\lambda_1 = 2$ and $\lambda_2 = 3$ as distinct real eigenvalues with eigenvector Y_1 and Y_2 respectively. Using Theorem 5.1, the third-order ODE (6.102) can be transformed to the canonical form of $L_{3,6}^{II}$ via a point transformation.

In order to construct such a point transformation, one needs to match the the two eigenvectors of $\text{ad}(G/G')$ on G' with $X = \frac{\partial}{\partial y}$ and $Y = x \frac{\partial}{\partial y}$ respectively, in the following way:

$$X = Y_1, \quad Y = Y_2. \quad (6.105)$$

Applying the correspondence (6.105) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_y = 0, \quad x\phi_y = 0, \quad \psi_y = 1, \quad x\psi_y = e^\phi. \quad (6.106)$$

The solution of the system (6.106) gives the following point transformation

$$u = \ln x, \quad v = y, \quad (6.107)$$

which transforms the vector $\frac{1}{2}Y_3$ which is linearly independent of G' to

$$Z = (c-1)x\frac{\partial}{\partial x} + (-y + f(x))\frac{\partial}{\partial y}, \quad (6.108)$$

with $f(x) = -2\ln x$ and $c = \frac{\lambda_2}{\lambda_1} = \frac{3}{2}$. The function $f(x)$ can be absorbed using the transformation

$$\tilde{x} = x, \quad \tilde{y} = y + \frac{1}{1-c} x^{\frac{1}{1-c}} \int f(x)x^{\frac{2-c}{c-1}} dx = y + 2\ln x - 1. \quad (6.109)$$

Finally, the composition of the transformations (6.107) and (6.109) transforms ODE (6.102) to its canonical form

$$\tilde{y}''' = \tilde{x}^{\frac{2-3c}{c-1}} g\left(\tilde{y}''\tilde{x}^{\frac{2c-1}{c-1}}\right), \quad c \neq 0, 1 \quad (6.110)$$

with $g(z) = \frac{1}{2}z^2$ and $c = \frac{3}{2}$.

Example 6.11. $L_{3:7}^I$

Consider the ODE

$$v'' = \frac{1}{9}(2v'^2 - 2v' + 5)^{\frac{3}{2}} \exp\left(3 \arctan\left(\frac{v' - 2}{v' + 1}\right)\right) \quad (6.111)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial u}, \quad Y_2 = \frac{\partial}{\partial v}, \quad Y_3 = (4u + v)\frac{\partial}{\partial u} + (5v - \frac{5}{2}u)\frac{\partial}{\partial v}, \quad (6.112)$$

with the nonzero commutators

$$[Y_1, Y_3] = 4Y_1 - \frac{5}{2}Y_2, \quad [Y_2, Y_3] = Y_1 + 5Y_2. \quad (6.113)$$

Here $\dim G' = 2$, $\text{rank}(G') = 2$ and the adjoint action of $G/G' = \langle \bar{Y}_3 \rangle$ on $G' = \langle Y_1, Y_2 \rangle$ is given by

$$\text{ad}(\bar{Y}_3) = \begin{pmatrix} -4 & -1 \\ \frac{5}{2} & -5 \end{pmatrix}. \quad (6.114)$$

The eigenvalues are $-\frac{9}{2} \pm \frac{3}{2}\sqrt{-1}$ with eigenvectors $(\frac{1}{5} \pm \frac{3}{5}\sqrt{-1})Y_1 + Y_2$ respectively. Using the Theorem 5.1, the second-order ODE (6.111) can be transformed to the canonical form of $L_{3:7}^I$ via a point transformation.

In order to construct such a point transformation, one needs to match the real and imaginary parts of an eigenvector of $\text{ad}(G/G')$ on G' with $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ respectively in the following way:

$$X = \frac{1}{5}Y_1 + Y_2, \quad Y = \frac{3}{5}Y_1. \quad (6.115)$$

Applying the correspondence (6.115) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_x = \frac{1}{5}, \quad \phi_y = \frac{3}{5}, \quad \psi_x = 1, \quad \psi_y = 0. \quad (6.116)$$

Solving the system (6.116) gives the following point transformation

$$u = \frac{1}{5}x + \frac{3}{5}y, \quad v = x, \quad (6.117)$$

which transforms ODE (6.111) after simplification using the identity

$$\tan^{-1}x - \tan^{-1}\left(\frac{2x-1}{x+2}\right) = c = \tan^{-1}\left(\frac{1}{2}\right)$$

to its canonical form

$$y'' = C(1 + y'^2)^{\frac{3}{2}} \exp(b \arctan y') \quad (6.118)$$

with $C = -\frac{e^{3c}}{\sqrt{5}}$ and $b = \cot \theta = -3$ where $\theta = \arg(-\frac{9}{2} + \frac{3}{2}\sqrt{-1})$.

Example 6.12. $L_{3:7}^{II}$

Consider the ODE

$$v''' = \frac{4e^{-8 \arctan(\frac{1}{u})}}{(u^2 + 1)^{\frac{5}{2}} u \left(1 - 4(u^2 + 1)^{\frac{3}{2}}(uv'' + 2v')\right)} - 3 \frac{2u^2v'' + 2v'u + v''}{(u^2 + 1)u} \quad (6.119)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial v}, \quad Y_2 = \frac{1}{u} \frac{\partial}{\partial v}, \quad Y_3 = (u^2 + 1) \frac{\partial}{\partial u} + \frac{1}{u} (4uv - v - \sqrt{u^2 + 1}) \frac{\partial}{\partial v}, \quad (6.120)$$

with the nonzero commutators

$$[Y_1, Y_3] = 4Y_1 - Y_2, \quad [Y_2, Y_3] = Y_1 + 4Y_2. \quad (6.121)$$

Here $\dim G' = 2$, $\text{rank}(G') = 1$ and the adjoint action of $G/G' = \langle \bar{Y}_3 \rangle$ on $G' = \langle Y_1, Y_2 \rangle$ is given by

$$\text{ad}(\bar{Y}_3) = \begin{pmatrix} -4 & -1 \\ 1 & -4 \end{pmatrix}. \quad (6.122)$$

The eigenvalues are $-4 \pm \sqrt{-1}$ with eigenvectors $Y_2 \pm \sqrt{-1} Y_1$ respectively. Using the Theorem 5.1, the third-order ODE (6.119) can be transformed to the canonical form of $L_{3:7}^{II}$ via a point transformation.

To construct such a point transformation, one needs to match the real part and imaginary part of an eigenvector of $\text{ad}(G/G')$ on G' with $X = \frac{\partial}{\partial y}$ and $Y = x \frac{\partial}{\partial y}$ in the following way:

$$X = Y_2, \quad Y = Y_1. \quad (6.123)$$

Applying the correspondence (6.123) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\phi_y = 0, \quad x\phi_y = 0, \quad \psi_y = \frac{1}{\phi}, \quad x\psi_y = 1. \quad (6.124)$$

Solving the system (6.124) gives the following point transformation

$$u = x, \quad v = \frac{y}{x}, \quad (6.125)$$

which transforms the vector Y_3 which is linearly independent of G' to

$$\left(\frac{1}{\sin\theta}\right)Z = (1 + x^2) \partial_x + (y(x - b) + f(x)) \partial_y, \quad (6.126)$$

with $f(x) = -\sqrt{x^2 + 1}$ and $b = \cot\theta = -4$. The function $f(x)$ can be absorbed using the transformation

$$\tilde{x} = x, \quad \tilde{y} = y - \sqrt{x^2 + 1} e^{-b \tan^{-1} x} \int \frac{e^{b \tan^{-1} x}}{(x^2 + 1)^{\frac{3}{2}}} f(x) dx = y - \frac{1}{4} \sqrt{x^2 + 1}. \quad (6.127)$$

Finally, the composition of the transformations (6.125) and (6.127) transforms ODE (6.119) after simplification using the identity $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$ to its canonical form

$$\tilde{y}''' = \frac{\tilde{y}''}{1 + \tilde{x}^2} \left(f \left(\tilde{y}'' (\tilde{x}^2 + 1)^{\frac{3}{2}} e^{b \tan^{-1} \tilde{x}} \right) - 3\tilde{x} \right) \quad (6.128)$$

with $f(z) = -\frac{e^{4\pi}}{z^2}$ and $b = -4$.

Example 6.13. $L_{3:8}^I$

Consider the ODE

$$v'' = (v + v')^3 - \frac{1}{2}v - \frac{3}{2}v' \quad (6.129)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial u}, \quad Y_2 = \exp(-u) \frac{\partial}{\partial v}, \quad Y_3 = v \exp(u) \frac{\partial}{\partial u} - \frac{1}{2}v^2 \exp(u) \frac{\partial}{\partial v}, \quad (6.130)$$

with the nonzero commutators

$$[Y_1, Y_2] = -Y_2, \quad [Y_1, Y_3] = Y_3, \quad [Y_2, Y_3] = Y_1. \quad (6.131)$$

Since $\dim G' = 3$, the Killing form is indefinite and $\text{rank } G = 2$, using the Theorem 5.1, the second-order ODE (6.129) can be transformed to one of the three canonical forms $L_{3:8}^I$, $L_{3:8}^{II}$ and $L_{3:8}^{III}$ via a point transformation.

Since the eigenvalues of $\text{ad}(Y_1)$ are ± 1 , then by a scaling, as explained in section 4.3, one can get the change of basis

$$X = Y_3, \quad Y = -2Y_2, \quad Z = 2Y_1. \quad (6.132)$$

This maps the nonzero commutators (6.131) to the standard relations given by

$$[Z, X] = 2X, \quad [Z, Y] = -2Y, \quad [X, Y] = Z.$$

Applying the correspondence (6.132) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\begin{aligned} \phi_y &= \exp(\phi)\psi, & \psi_y &= -\frac{1}{2}\exp(\phi)\psi^2, \\ -2xy\phi_x + (\epsilon x^2 - y^2)\phi_y &= 0, & -2xy\psi_x + (\epsilon x^2 - y^2)\psi_y &= -2\exp(-\phi), \\ x\phi_x + y\phi_y &= -1, & x\psi_x + y\psi_y &= 0, \end{aligned} \quad (6.133)$$

for some $\epsilon \in \{0, 1, -1\}$. Since the system (6.133) is consistent for $\epsilon = 0$, its solution

$$u = \ln\left(\frac{x}{y^2}\right) + c_1, \quad v = -2e^{-c_1}\left(\frac{y}{x}\right), \quad (6.134)$$

transforms ODE (6.129) to the canonical form $L_{3:8}^I$

$$xy'' = Cy'^3 - \frac{1}{2}y', \quad (6.135)$$

with $C = 4e^{-2c_1}$.

Example 6.14. $L_{3:8}^{II}$

Consider the ODE

$$u^4vv'' = (v'^2u^4 + 1)^{\frac{3}{2}} - 1 - 2vv'u^3 - v'^2u^4 \quad (6.136)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = u^2\frac{\partial}{\partial u}, \quad Y_2 = -u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad Y_3 = (u^2v^2 - 1)\frac{\partial}{\partial u} + \frac{2v}{u}\frac{\partial}{\partial v}, \quad (6.137)$$

with the nonzero commutators

$$[Y_1, Y_2] = Y_1, \quad [Y_1, Y_3] = -2Y_2, \quad [Y_2, Y_3] = Y_3. \quad (6.138)$$

Since $\dim G' = 3$, the Killing form is indefinite and $\text{rank } G = 2$, using the Theorem 5.1, the second-order ODE (6.136) can be transformed to one of the three canonical forms $L_{3:8}^I$, $L_{3:8}^{II}$ and $L_{3:8}^{III}$ via a point transformation.

Since the eigenvalues of $\text{ad}(Y_2)$ are ± 1 , by a scaling, as explained in section 4.3, one can get the change of basis

$$X = Y_3, \quad Y = Y_1, \quad Z = 2Y_2. \quad (6.139)$$

This maps the nonzero commutators (6.138) to the standard relations given by

$$[Z, X] = 2X, [Z, Y] = -2Y, [X, Y] = Z.$$

Applying the correspondence (6.139) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\begin{aligned} \phi_y &= \phi^2 \psi^2 - 1, & \psi_y &= \frac{2\psi}{\phi}, \\ -2xy\phi_x + (\epsilon x^2 - y^2)\phi_y &= \phi^2, & -2xy\psi_x + (\epsilon x^2 - y^2)\psi_y &= 0, \\ x\phi_x + y\phi_y &= \phi, & x\psi_x + y\psi_y &= -\psi, \end{aligned} \quad (6.140)$$

for some $\epsilon \in \{0, 1, -1\}$. Since the system (6.140) is consistent for $\epsilon = 1$, its solution

$$u = -\frac{x^2+y^2}{y}, \quad v = \frac{x}{x^2+y^2}, \quad (6.141)$$

transforms ODE (6.136) to the canonical form $L_{3:8}^{II}$

$$xy'' = y' + y'^3 + C(1 + y'^2)^{\frac{3}{2}}, \quad (6.142)$$

with $C = 1$.

Example 6.15. $L_{3:8}^{III}$

Consider the ODE

$$u^4 v^3 v'' = 3v^2 v'^2 u^4 - v^6 - 2v^3 v' u^3 + (v'^2 u^4 - v^4)^{\frac{3}{2}} \quad (6.143)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = u^2 \frac{\partial}{\partial u}, \quad Y_2 = -u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}, \quad Y_3 = \left(1 + \frac{u^2}{v^2}\right) \frac{\partial}{\partial u} + \frac{2v}{u} \frac{\partial}{\partial v}, \quad (6.144)$$

with the nonzero commutators

$$[Y_1, Y_2] = Y_1, \quad [Y_1, Y_3] = 2Y_2, \quad [Y_2, Y_3] = Y_3. \quad (6.145)$$

Here again the Killing form is non-degenerate and indefinite and rank $G = 2$. Using the Theorem 5.1, the second-order ODE (6.143) can be transformed to one of the three canonical forms $L_{3:8}^I$, $L_{3:8}^{II}$ and $L_{3:8}^{III}$ via a point transformation.

Since the eigenvalues of $\text{ad}(Y_2)$ are ± 1 , then by a scaling, as explained in section 4.3, one has the change of basis

$$X = Y_3, \quad Y = -Y_1, \quad Z = 2Y_2. \quad (6.146)$$

This maps the nonzero commutators (6.145) to the standard relations given by

$$[Z, X] = 2X, [Z, Y] = -2Y, [X, Y] = Z.$$

Applying the correspondence (6.146) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\begin{aligned} \phi_y &= 1 + \frac{\phi^2}{y^2}, & \psi_y &= \frac{2\psi}{\phi}, \\ -2xy\phi_x + (\epsilon x^2 - y^2)\phi_y &= -\phi^2, & -2xy\psi_x + (\epsilon x^2 - y^2)\psi_y &= 0, \\ x\phi_x + y\phi_y &= \phi, & x\psi_x + y\psi_y &= \psi, \end{aligned} \quad (6.147)$$

for some $\epsilon \in \{0, 1, -1\}$. Since the system (6.147) is consistent for $\epsilon = -1$, its solution

$$u = \frac{y^2 - x^2}{y}, \quad v = \frac{y^2 - x^2}{x}, \quad (6.148)$$

transforms ODE (6.143) to the canonical form $L_{3:8}^{III}$

$$xy'' = y' - y'^3 + C(1 - y'^2)^{\frac{3}{2}}, \quad (6.149)$$

with $C = -1$.

Example 6.16. $L_{3:8}^{IV}$

Consider the ODE

$$v''' = \frac{3v''^2}{2v'} - \frac{v'^3}{v^2} \quad (6.150)$$

that admits the three-dimensional point symmetry **subalgebra** generated by

$$Y_1 = u \frac{\partial}{\partial u}, \quad Y_2 = \frac{\partial}{\partial u}, \quad Y_3 = \frac{1}{2}u^2 \frac{\partial}{\partial u}, \quad (6.151)$$

with the nonzero commutators

$$[Y_1, Y_2] = -Y_2, \quad [Y_1, Y_3] = Y_3, \quad [Y_2, Y_3] = Y_1. \quad (6.152)$$

Here $\dim G' = 3$, the Killing form is indefinite and $\text{rank } G' = 1$. Using the Theorem 5.1, the third-order ODE (6.150) can be transformed to the canonical form $L_{3:8}^{IV}$ via a point transformation.

Since the eigenvalues of $\text{ad}(Y_1)$ are ± 1 , then by a scaling, as explained in section 4.3, one can get the change of basis

$$X = Y_3, \quad Y = -2Y_2, \quad Z = 2Y_1. \quad (6.153)$$

This maps the nonzero commutators (6.131) to the standard relations given by

$$[Z, X] = 2X, \quad [Z, Y] = -2Y, \quad [X, Y] = Z.$$

Applying the correspondence (6.153) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\begin{aligned} \phi_y &= \frac{1}{2}\phi^2, & \psi_y &= 0, \\ -y^2\phi_y &= -2, & -y^2\psi_y &= 0, \\ y\phi_y &= -\phi, & y\psi_y &= 0. \end{aligned} \quad (6.154)$$

The solution of the system (6.154) gives a point transformation

$$u = -\frac{2}{y}, \quad v = x, \quad (6.155)$$

that transforms ODE (6.150) to the canonical form $L_{3:8}^{IV}$

$$y''' = \frac{3}{2} \frac{y''^2}{y'} + f(x)y', \quad (6.156)$$

with $f(x) = \frac{1}{x^2}$.

Example 6.17. $L_{3:9}$

Consider the ODE

$$v'' = -v'^3 \cos u \sin u - 2v' \cot u + \csc u (v'^2 \sin^2 u + 1)^{\frac{3}{2}} \quad (6.157)$$

that admits the three-dimensional point symmetry algebra generated by

$$Y_1 = \frac{\partial}{\partial v}, \quad Y_2 = \sin v \frac{\partial}{\partial u} + \cos v \cot u \frac{\partial}{\partial v}, \quad Y_3 = \cos v \frac{\partial}{\partial u} - \sin v \cot u \frac{\partial}{\partial v}, \quad (6.158)$$

with the nonzero commutators

$$[Y_1, Y_2] = Y_3, \quad [Y_1, Y_3] = -Y_2, \quad [Y_2, Y_3] = Y_1. \quad (6.159)$$

Since the Killing form is negative definite, using Theorem 5.1, the second-order ODE (6.157) can be transformed to the canonical form of $L_{3:9}$ via a point transformation.

In order to construct such a point transformation, pick any vector like Y_1 and find its non-zero eigenvalues. Here $\text{ad}(Y_1)$ has $\pm\sqrt{-1}$ as eigenvalues with eigenvectors $Y_3 \pm \sqrt{-1} Y_2$ respectively. One needs to match the vector Y_1 with X and a multiple of eigenvector $Y_3 + \sqrt{-1} Y_2$ with the vector $Y - \sqrt{-1} Z$ such that $[Y, Z] = X$ in the following way:

$$X = Y_1, \quad Y = Y_3, \quad Z = -Y_2. \quad (6.160)$$

Applying the correspondence (6.160) to the point transformation $u = \phi(x, y), v = \psi(x, y)$ yields the system

$$\begin{aligned} \phi_x &= 0, & \psi_x &= 1, \\ y \sin x \phi_x + (y^2 + 1) \cos x \phi_y &= \cos \psi, & y \sin x \psi_x + (y^2 + 1) \cos x \psi_y &= -\sin \psi \cot \phi, \\ y \cos x \phi_x - (y^2 + 1) \sin x \phi_y &= -\sin \psi, & y \cos x \psi_x - (y^2 + 1) \sin x \psi_y &= -\cos \psi \cot \phi. \end{aligned} \quad (6.161)$$

Solution of the system (6.161) gives the required point transformation. A systematic way of solving such a nonlinear system is as follows:

One can match the vector Y_1 with X through the canonical coordinates of Y_1 as

$$u = y, \quad v = x. \quad (6.162)$$

This transforms the vector $Y - \sqrt{-1} Z$ to

$$Y - \sqrt{-1} Z = Y_3 + \sqrt{-1} Y_2 = e^{\sqrt{-1} x} [(f_1(y)\partial_x + f_2(y)\partial_y) + \sqrt{-1}(f_3(y)\partial_x + f_4(y)\partial_y)], \quad (6.163)$$

with $f_1(y) = 0, f_2(y) = 1, f_3(y) = \cot y$ and $f_4(y) = 0$. Now using the formula (4.23), the vector $Y - \sqrt{-1} Z$ can be transformed using the transformation

$$\tilde{x} = x + \tan^{-1} \left(\frac{f_4}{f_2} \right) = x, \quad \tilde{y} = \frac{f_1 f_4 - f_2 f_3}{\sqrt{f_2^2 + f_4^2}} = -\cot y. \quad (6.164)$$

to the canonical form

$$Y - \sqrt{-1} Z = e^{\sqrt{-1}\tilde{x}} [(1 + \tilde{y}^2)\partial_{\tilde{y}} - \sqrt{-1} \tilde{y}\partial_{\tilde{x}}], \quad (6.165)$$

Finally, the composition of the transformations (6.162) and (6.164) given by

$$u = -\cot^{-1} \tilde{y}, \quad v = \tilde{x}, \quad (6.166)$$

transforms ODE (6.157) to its canonical form

$$\tilde{y}'' = C \left(\frac{\tilde{y}'^2 + \tilde{y}^2 + 1}{1 + \tilde{y}^2} \right)^{\frac{3}{2}} - \tilde{y} \quad (6.167)$$

with $C = 1$.

Moreover, a solution of the nonlinear system (6.161) is

$$u = -\cot^{-1} y, \quad v = x. \quad (6.168)$$

7 Conclusion

The Lie-Bianchi classification of three-dimensional algebras and their realizations as vector fields in \mathbb{R}^2 are recovered in an algorithmic way. This is done in such a way that one can read off the type of the algebra from its invariants like the dimension of its commutator or the centralizer of its commutator and its rank.

The compact and non-compact Lie algebras are treated uniformly in a manner that makes their realizations as vector fields in the plane transparent.

The algorithms are illustrated by examples for each type of three dimensional algebras. The procedure works in principle for any ODE which admits a three-dimensional subalgebra of symmetries.

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