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Abstract. Algorithms for embedding certain types of nilpotent subalgebras in maximal subalgebras of the same type are developed, using methods of real algebraic groups. These algorithms are applied to determine non-conjugate subalgebras of the symmetry algebra of the wave equation, which in turn are used to determine a large class of invariant solutions of the wave equation. The algorithms are also illustrated for the symmetry algebra of a classical system of differential equations considered by Cartan in the context of contact geometry.

1. Introduction

The principal aim of this note is to give algorithms for embedding abelian and solvable algebras of certain types in maximal subalgebras of the same type, using standard commands of Maple.

These algorithms are useful in finding reductions given by nonconjugate classes of subalgebras of symmetries of differential equations as well as their invariant solutions; see [Ib1, Ch 9, p.224].

The precise types of the subalgebras are given in the algorithms given below. The algorithms also give both the relative and absolute root system of the Lie algebra more or less automatically.

Maple is able to find the Cartan decomposition as well as root space decompositions for semisimple algebras of fairly high dimensions. The algorithms it uses are based on the fundamental papers of Rand, Winternitz and Zassenhuas [RWZ] and of deGraff [dG]. For the reduced root space decomposition it needs a maximal split abelian algebra of semisimple elements. If that is not specified, the commands will give, in general, roots taking complex eigenvalues.

The algorithms given in this paper are based on results of Mostow [Mo] on real algebraic groups; see a recent account of the subject in [AB]. The Lie algebras occurring here include all semisimple Lie algebras, all linear Lie algebras generated by nilpotents, all abelian linear algebras of semisimple elements of Lie algebras defined by integral equations as well as all subalgebras generated by the types of algebras already listed.

We will illustrate the algorithms by working out in detail an embedding of a subalgebra, that is clearly abelian, of the symmetry algebra of the wave equation on flat 4 dimensional space, which gives at the same time detailed structure of the symmetry algebra and several nonconjugate subalgebras. We do the same for the symmetry algebra of a classical nonlinear ODE, whose symmetry algebra was determined and identified as the exceptional algebra G_2 by Cartan – see [Ag] for further references, and also [AKO].

The structural information obtained gives several three and four dimensional subalgebras that are nonconjugate in the adjoint representation. This gives a more extensive and useable list of subalgebras than that given in [ADGM] and these subalgebras are used in the last section to give solutions of the wave equation in flat 4 dimensional space.

In a follow up of this paper, a similar analysis of the wave equation on all static spherically symmetric spaces times will be given to obtain a more extensive list of solutions than that given

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in [ADGM] of the wave equation on certain spherically symmetric four dimensional spaces. In particular, such invariant solutions will be given for all types of three dimensional subalgebras that can arise as subalgebras of the symmetry algebra of the equation.

The reader is referred to Ibragimov [Ib2] for very general results on the wave equation on Riemannian manifolds and several other equations of relevance to physics.

As far as Lie algebras are concerned, we need the following results:

Proposition 1. *Let X be an element of a Lie algebra L which is the Lie algebra of a real algebraic subgroup of* $\mathrm{GL}(n,\mathbb{R})$. *Let $X = X_s + X_n$ be the Jordan decomposition of X. Then both X_s and X_n lie in L. In fact, they are in the center of the centralizer of X in L.*

If furthermore $AX_sA^{-1} = U + \sqrt{-1}V$, where U and V are real diagonal matrices, then both $A^{-1}UA$ and $A^{-1}\sqrt{-1}V A$ lie in L.

Moreover, if H is any subalgebra of L then the derived algebras of the radical of centralizer of H and of the normalizer of H are ad-nilpotent.

If $H = K + P$ is the Cartan decomposition then we call K the polar part of H and P the radial part of H. For p in P all the eigenvalues of ad p are real and if u is an eigenvector of ad p corresponding to a non-zero eigenvalue then ad u is nilpotent

Also if H is a semisimple subalgebra of L and X is an ad-semisimple or ad-nilpotent element of H in the adjoint representation of H on itself, then it is also ad-semisimple or ad-nilpotent in the adjoint representation of L.

These facts are very useful in verifying that a certain element is semisimple or nilpotent by reducing the computations to subalgebras of small dimensions. We begin with some basic concepts and recollections. For more details the reader is referred to [HN], [Kn].

A few words regarding the section on roots is in order. Maple will give – for any Cartan algebra – an array of complex numbers. In the following section we explain how to extract a simple system of roots and the corresponding Dynkin diagram directly from such a list.

2. Roots

2.1. **Roots of a semisimple algebra.** Let L be a semisimple Lie algebra of $\mathfrak{g}\ell(n,\mathbb{R})$ and C a Cartan subalgebra of L. The algebra C is, by definition, a maximal abelian subalgebra of diagonalizable elements in the complexification of L. A nonzero vector v in $L\oplus\sqrt{-1}L$ such that

$$[h\,,v] \,=\, \lambda(h)\cdot v$$

for all $h \in C$ is called a *root vector* and the corresponding linear functional λ is called a root of the Cartan algebra C.

In general, the roots will be complex valued, so one needs to clarify what it means for a complex valued root to be positive – based only on the list of roots provided by the program. This is sufficient to describe the Dynkin diagram algorithmically, as detailed below.

A complex number $z = a + \sqrt{-1}b$, where $a\,,b \in \mathbb{R}$, is positive if either its real part a is positive or $a = 0$ but $b > 0$.

Fix a basis $h_1\,,\cdots,h_r$ of C. A non-zero root λ is positive if the first nonzero number $\lambda(h_i)$ is a complex positive number. Otherwise, it is called a negative root.

For sake of convenience, henceforth a root will mean a non-zero root.

2.2. **Restricted roots.** An abelian subalgebra of L consisting of semisimple elements in the adjoint representation on L, is, by definition, a *torus*. If, moreover, all its elements in the adjoint representation of L have real eigenvalues, then it is a *real* torus; if all eigenvalues are purely imaginary, it is called a *compact* torus.

All maximal solvable subalgebras B of a real semisimple algebra with real eigenvalues in the adjoint representation are conjugate [Mo], [AB].

If A is a maximal torus of B, then the roots in B are real valued. Roots in B which are not a sum of two roots in B are simple roots of a root system – in the sense of [HN], [Kn].

2.3. **Procedure for constructing Dynkin diagram.** Positive roots which are not a sum of two positive roots are called simple roots. Thus to obtain simple roots from a given set of positive roots, one adds pairs of positive roots and marks those that are sums of positive roots; at the end, one strikes out those roots that are sums of two positive roots and the remaining ones will be simple roots.

Let a, b be simple roots. The positive roots among the integral combinations of them determine the bond between a and b. The simple roots a, b are not joined if $a+b$ is not a root. They are joined by a single bond if a, b and $a+b$ are the only positive roots among the integral combinations of a and b. They are joined by a double bond with arrow pointing from a to b if a, b, $a+b$ and $a+2b$ are the only positive roots among the integral combinations of a and b. They are joined by a triple bond with arrow pointing from a to b if a, b, $a+b$, $a+2b$, $a+3b$ and $2a+3b$ are the only positive roots among the integral combinations of a and b.

The diagram then identifies the complexification of the Lie algebra L.

3. Algorithms

The algorithms given below are similar to each other. For the convenience of the user we have written down complete details-at the expense of repetition-of the most frequent types of algebras encountered in practice.

3.1. **Algorithm for embedding a given abelian subalgebra of semisimple elements with real eigenvalues in a maximal algebra of such elements and in a maximally real Cartan algebra.** For a subalgebra H of L, let $N_L(H)$, $Z_L(H)$, $Z(H)$ and H' denote its normalizer in L, its centralizer in L, its center and its derived algebra respectively. For notational convenience, we will also write $N(H)$ for $N_L(H)$.

Let A be real torus (defined in Section 2.2).

Step 1: Compute $Z_L(A)$, the centralizer of A in L, the derived algebra $Z_L(A)'$ of $Z_L(A)$ and the center $Z(Z_L(A))$ of $Z_L(A)$. Then one has the direct sum decomposition

$$Z_L(A) \,=\, Z_L(A)' \oplus Z(Z_L(A))\,.$$

Step 2: Compute the Killing form of $Z_L(A)'$. If it is negative definite, then the real part of the subalgebra $Z(Z_L(A))$ is a maximal real torus.

Step 3: If the Killing form of $Z_L(A)'$ is indefinite, compute the Cartan decomposition of $Z_L(A)'$, and pick any nonzero element from the radial part of the decomposition and adjoin it to A.

Repeat Step 1 and Step 2 till an abelian algebra, which we again denote by A, is obtained which has all real eigenvalues in the adjoint representation – and in the decomposition $Z_L(A) = Z_L(A)' \oplus Z(Z_L(A))$, the Killing form of $Z_L(A)'$ is negative definite.

At this stage, a maximal real torus containing the given algebra has been obtained. Denote it again by A.

The compact part of $Z(Z_L(A))$ together with a maximal torus of $Z_L(A)'$ is a compact torus. Adjoining it to A gives a maximally real Cartan algebra.

Remark 2. By an entirely similar procedure, a compact torus can be embedded in a maximally compact Cartan subalgebra.

3.2. **Algorithm for embedding a commutative subalgebra of ad-nilpotent elements to a maximal commutative subalgebra of such elements.** Let U be an abelian algebra of ad–nilpotent elements. Let

$$Z_L(U) \,=\, S \oplus R$$

be the Levi decomposition of $Z_L(U)$. Compute the derived subalgebra $R' \,\subset\, R$. If $\dim(R'+U) > \dim U$, adjoin any element of R' to U to obtain a commutative subalgebra of ad–nilpotent elements. Repeat this procedure until an abelian algebra of ad–nilpotent elements is obtained – which we denote again by U – so that in the Levi decomposition on $Z_L(U)$, the algebra R' is contained in U.

At this stage if R contains U as a proper subalgebra, consider an element of x of R complementary to U. Then the nilpotent and semisimple parts of x belong to R. If x has a nonzero nilpotent part x_n, then the subalgebra generated by U and x_n is commutative consisting of ad–nilpotent elements.

Repeating the above procedure, we may assume that U is a commutative subalgebra of ad–nilpotent elements such that in the Levi decomposition

$$Z_L(U) \,=\, S \oplus R\,,$$

$R' \,\subset\, U$, and every element in a basis of R complementary to U consists of semisimple elements.

If the Killing form of S is not negative definite, then S will have a nontrivial Cartan decomposition. Take an element α in the radial part of the Cartan decomposition of S. As S has no center, the endomorphism $\mathrm{ad}(\alpha)$ of S has a nonzero real eigenvalue. In fact any element all of whose eigenvalues are real will do. If u is a nonzero eigenvector of $\mathrm{ad}(\alpha)$ for such a nonzero real eigenvalue, then $\mathrm{ad}(u)$ is nilpotent on S and therefore on the Lie algebra L – by uniqueness of the Jordan decomposition. Adjoin u to U to obtain a higher dimensional commutative algebra of nilpotents.

Thus repeating the above procedure we ultimately have a commutative subalgebra consisting of ad–nilpotent elements, which we again denote by U, such that

$$Z_L(U) \,=\, S + R\,,$$

$R' \,\subset\, U$, every element in a basis of R complementary to U consists of semisimple elements and S has a negative definite Killing form.

At this stage U is a maximal abelian subalgebra of ad–nilpotent elements.

A very similar argument (-detailed below-) gives the embedding of a given ad–nilpotent subalgebra in a maximal ad–nilpotent subalgebra. Here one is assured of conjugacy of these subalgebras [AB]. Also, its normalizer will pick up a torus whose eigenvalues are all real and which is maximal with these properties. Embedding this in a maximal torus – using Algorithm 1 – one will obtain a maximally split Cartan subalgebra of L.

3.3. **Algorithm for embedding a subalgebra of ad–nilpotent elements to a maximal subalgebra of such elements. Step 1:** Find a normalizer of U and compute its Levi decomposition

$$N(U) \,=\, S \oplus R\,,$$

where R is the radical.

Step 2: Compute the derived algebra R' of R. If $\dim(R'+U) > \dim U$, then this is again an ad–nilpotent algebra.

Repeat Step 1 and Step 2 so that ultimately $R'+U = U$. At this stage R/U is abelian.

Step 3: We want to enlarge U further so that R/U consists entirely of semisimple elements. To do this, take a basis of U, say $u_1, \cdots, u_k$, and enlarge it to a basis of R by adjoining $v_1, \cdots, v_\ell$. Find the Jordan decomposition of all $v_1, \cdots, v_\ell$. Adjoin to U the nilpotent parts of all the $v_1, \cdots, v_\ell$. Denote by $\widetilde{U}$ the algebra obtained this way.

Now repeat Step 1, Step 2 and Step 3 till we have the Levi decomposition

$$N(\widetilde{U}) = S + R$$

such that $R' \subset U$ and $R/\widetilde{U}$ consists only of semisimple elements.

Step 4: If the Killing form of S is not negative definite, then it will have a nontrivial Cartan decomposition. Take an element α in the radial part of the Cartan decomposition of S. As the center of S is trivial, the endomorphism $\mathrm{ad}(\alpha)$ of S has a nonzero real eigenvalue. Let u be a nonzero eigenvector for such an eigenvalue. Then $\mathrm{ad}(u)$ is nilpotent on S and therefore on the Lie algebra L – by uniqueness of the Jordan decomposition. Adjoin u to U to obtain a higher dimensional algebra of nilpotents.

Iterating this procedure, we finally reach the situation that we have an ad–nilpotent subalgebra, which we denote again by U, that contains the original ad–nilpotent subalgebra, such that the Levi decomposition of $N(U)$ is

$$N(U) = S \oplus R,$$

where S has a negative definite Killing form, $R' \subset U$ and R/U consists only of semisimple elements in the sense that if we extend a basis of U to a basis of R and find the Jordan decomposition of the basis elements outside U, then the nilpotent parts all belong to U [AB].

At this stage, U is a maximal ad–nilpotent algebra containing the given ad–nilpotent algebra.

Finally, the abelian algebra representing R/U is a torus and its real part is a maximal abelian algebra consisting of real semisimple elements [AB]. Denote this algebra by A. Then A can be enlarged to a maximally split Cartan algebra of the whole algebra, using Algorithm 1. This algebra permutes the common eigenspaces of A.

4. Applications to structure of symmetry algebras related to certain equations of physics

Before giving applications to symmetry algebras of higher dimensions, we illustrate the algorithms to obtain structural information for the algebras $\mathfrak{so}(4)$, $\mathfrak{so}(1,3)$ and $\mathfrak{so}(2,2)$.

Recall that if A is an $n \times n$ diagonal matrix with diagonal entries $1, -1$, then the Lie algebra of the corresponding orthogonal group has generators $e_{ij} - e_{ji}$ if $a_{ii}a_{jj} = 1$, and $e_{ij} + e_{ji}$ if $a_{ii}a_{jj} = -1$. Moreover, if A has the first p diagonal entries 1, and the next q diagonal entries -1, where $p+q = n$, then the Lie algebra of the corresponding orthogonal group is generated by

$$e_{1,2} - e_{2,1}, \cdots, e_{p-1,p} - e_{p,p-1}, e_{p,p+1} + e_{p+1,p}, e_{p+1,p+2} - e_{p+2,p+1}, \cdots, e_{n-1,n} - e_{n,n-1}.$$

4.1. $\mathfrak{so}(4)$.

We now derive the factorization of $\mathfrak{so}(4)$ using roots of its maximal torus.

A basis of $V = \mathfrak{so}(4)$ is

$$e_1 = e_{12} - e_{21}, e_2 = e_{13} - e_{31}, e_3 = e_{14} - e_{41}, e_4 = e_{23} - e_{32}, e_5 = e_{24} - e_{42}, e_6 = e_{34} - e_{43}.$$

Using Algorithm 1, a Cartan subalgebra C is generated by $\{e_1, e_6\}$. The roots of C are

$$a := (\sqrt{-1}, \sqrt{-1}), b := (\sqrt{-1}, -\sqrt{-1}), -a, -b.$$

As $a+b$ is not a root, this root system is of type $A_1 \times A_1$.

Also, conjugation maps a root to its negative. Thus the subalgebras generated by the eigenspaces V_r, V_{-r}, $r = a, b$, contain a real form of $\mathrm{sl}(2,\mathbb{C})$, which must be isomorphic to $\mathfrak{so}(3)$.

In more detail, we need to compute only the eigenvectors for the positive roots. Their real and imaginary parts will give the decomposition of the compact algebra V. Now,

$$V_a = \langle e_2 + \sqrt{-1}e_3 + \sqrt{-1}e_4 - e_5 \rangle, V_b = \langle e_2 + \sqrt{-1}e_3 - \sqrt{-1}e_4 + e_5 \rangle.$$

The real and imaginary parts of the basis elements in V_a and V_b and generate $\mathfrak{so}(4)$.

Let

$$u_1 = e_2 - e_5, v_1 = e_3 + e_4, u_2 = e_2 + e_5, v_2 = e_3 - e_4.$$

Then $u_1, v_1, [u_1, v_1] = -2(e_1 + e_6)$ generate a copy of $\mathfrak{so}(3)$. Note that $u_2, v_2, [u_2, v_2] = 2(e_1 - e_6)$ also generate a copy of $\mathfrak{so}(3)$. These two copies of $\mathfrak{so}(3)$ commute because the root system is of type $A_1 \times A_1$. This gives the well known fact that $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

4.2. $\mathfrak{so}(1,3)$**.** A basis of $V = \mathfrak{so}(1,3)$ is

$$e_1 = e_{12} + e_{21}, e_2 = e_{13} + e_{31}, e_3 = e_{14} + e_{41}, e_4 = e_{23} - e_{32}, e_5 = e_{24} - e_{42}, e_6 = e_{34} - e_{43}.$$

Using Algorithm 1 a Cartan subalgebra $C = \langle e_1, e_6 \rangle$ is obtained. Note that there is no real split or compact Cartan subalgebra.

The roots of C are

$$a := (1, -\sqrt{-1}), b := (1, \sqrt{-1}), -a, -b.$$

The root system is of type $A_1 \times A_1$, with positive roots a, b and conjugation maps a to b.

The real rank is one, and the eigenvalues of $\mathrm{ad}(e_1)$ are $-1, -1, 0, 0, 1, 1$. The corresponding root spaces are

$$V_1 = \langle e_3 + e_5, e_2 + e_4 \rangle, V_{-1} = \langle -e_3 + e_5, -e_2 + e_4 \rangle.$$

Also, $[e_3 + e_5, -e_3 + e_5] = 2e_1$; the subalgebra generated by $e_3 + e_5, -e_3 + e_5$ is $\mathrm{sl}(2,\mathbb{R})$, while the subalgebra generated by e_4, e_5, e_6 is $\mathfrak{so}(3)$.

Thus a maximal solvable subalgebra consisting of elements with real eigenvalues in the adjoint representation is

$$\langle e_1, e_3 + e_5, e_2 + e_4 \rangle,$$

and a maximal solvable subalgebra is

$$\langle e_6, e_1, e_3 + e_5, e_2 + e_4 \rangle.$$

4.3. $\mathfrak{so}(2,2)$**.** A basis of $V = \mathfrak{so}(2,2)$ is

$$e_1 = e_{12} - e_{21}, e_2 = e_{13} + e_{31}, e_3 = e_{14} + e_{41}, e_4 = e_{23} + e_{32}, e_5 = e_{24} + e_{42}, e_6 = e_{34} - e_{43}.$$

Using Algorithm 1, a real split Cartan subalgebra is $C = \langle e_2, e_5 \rangle$, while a compact Cartan subalgebra is $\langle e_1, e_6 \rangle$.

The roots of C are

$$a := (1, 1), b := (1, -1), -a, -b.$$

The root spaces are

$$V_a = \langle e_1 - e_3 + e_4 - e_6 \rangle, V_b = \langle e_1 + e_3 + e_4 + e_6 \rangle$$

$$V_{-a} = \langle e_1 + e_3 - e_4 - e_6 \rangle, V_{-b} = \langle e_1 - e_3 - e_4 + e_6 \rangle.$$

Conjugation fixes the roots. Consequently, the subalgebra generated by a root spaces of a root and its negative is isomorphic to $\mathrm{sl}(2,\mathbb{R})$.

Therefore, denoting the subalgebra generated by V_r, V_{-r} by $\langle V_r, V_{-r}\rangle$ the decomposition $\langle V_a, V_{-a}\rangle \oplus \langle V_b, V_{-b}\rangle$ gives an isomorphism of $\mathrm{sl}(2,\mathbb{R}) \oplus \mathrm{sl}(2,\mathbb{R})$ with $\mathfrak{so}(2,2)$.

4.4. **Lie symmetries of wave equations.** The algebra of Lie point symmetries of the wave equation in a flat 4-d space is sixteen dimensional and determined by the vector fields (following the same order as given in [ADGM] and using the notation in which X is represented with e):

$$\begin{aligned}
e_1 &= yt\partial_t + xy\partial_x + \frac{(y^2+t^2-x^2-z^2)}{2}\partial_y + yz\partial_z - uy\partial_u,\\
e_2 &= y\partial_t + t\partial_y,\\
e_3 &= xt\partial_t + \frac{(x^2+t^2-y^2-z^2)}{2}\partial_x + xy\partial_y + xz\partial_z - ux\partial_u,\\
e_4 &= x\partial_t + t\partial_x,\\
e_5 &= zt\partial_t + zx\partial_x + yz\partial_y + \frac{(z^2+t^2-y^2-x^2)}{2}\partial_z - uz\partial_u,\\
e_6 &= z\partial_t + t\partial_z,\\
e_7 &= t\partial_t + x\partial_x + y\partial_y + z\partial_z,\\
e_8 &= \partial_t,\\
e_9 &= (t^2+x^2+y^2+z^2)\partial_t + 2tx\partial_x + 2ty\partial_y + 2tz\partial_z - 2ut\partial_u,\\
e_{10} &= \partial_y,\\
e_{11} &= \partial_x,\\
e_{12} &= \partial_z,\\
e_{13} &= z\partial_y - y\partial_z,\\
e_{14} &= z\partial_x - x\partial_z,\\
e_{15} &= y\partial_x - x\partial_y,\\
e_{16} &= u\partial_u.
\end{aligned}$$

The commutator algebra of the finite dimensional part of the symmetry algebra of the wave equation on Minkowski space-time is 15 dimensional.

Its basis is $e_1, \ldots, e_6$, $e_7 - e_{16}, e_8, \ldots, e_{15}$. For notational convenience we will denote-only in this section- $e_7 - e_{16}$ by e_7.

The commutator table is reproduced in Appendix 1. The translations parallel to the coordinate axes are

$$\partial_x = e_{11}\,, \partial_y = e_{10}\,, \partial_z = e_{12}\,, \partial_t = e_8$$

and they form an ad–nilpotent subalgebra.

Let

$$U = \langle e_8\,, e_{10}\,, e_{11}\,, e_{12}\rangle\,.$$

We will use standard Maple commands and Algorithm 3 to embed U in a maximal ad-nilpotent subalgebra $\widetilde{U}$ and also compute the normalizer of this subalgebra.

As explained in Algorithm 3, in general, if

$$N(\widetilde{U}) = S + R$$

is the Levi decomposition of $N(\widetilde{U})$ then S has a negative definite Killing form, R' is contained in U and R/U consists only of semisimple elements in the sense that if we extend a basis of U to a basis of R and find the Jordan decomposition of the basis elements outside U, then the nilpotent parts all belong to U.

As the abelian algebra representing R/U is a torus and its real part A is a maximal abelian algebra consisting of real semisimple elements, A can be enlarged to a maximally split Cartan algebra of the whole algebra, using Algorithm 1. This algebra permutes the common eigenspaces of A.

Following Algorithm 3, we first compute $N(U)$ and its Levi decomposition. We have

$$N(U) = R \oplus S,$$

where $R = \langle U, e_7 \rangle$ is the radical, and

$$S = \langle e_{15}, e_{14}, e_{13}, e_6, e_5, e_4 \rangle$$

is semisimple with Cartan decomposition

$$S = \langle e_{15}, e_{14}, e_{13} \rangle \oplus \langle e_6, e_5, e_4 \rangle$$

with compact part $K = \langle e_{15}, e_{14}, e_{13} \rangle$ and radial part $P = \langle e_6, e_5, e_4 \rangle$.

The element e_7 representing R/U is real semisimple and e_6 is maximal abelian in P. The compact subalgebra K is the subalgebra of spatial rotations.

The eigenvalues of $\mathrm{ad}(e_6)$ in S, counting multiplicities, are $1, 1, -1, -1, 0, 0$ and eigenvectors for eigenvalue 1 are $-e_{15} + e_4$, $-e_{13} + e_6$. Therefore, as the eigenvectors for positive eigenvalues of a real semisimple element of S form an ad-nilpotent subalgebra, following Algorithm 3, we adjoin $-e_{15} + e_4$, $-e_{13} + e_6$ to U to get an ad-nilpotent algebra $\tilde{U}$ and compute its normalizer. We find that

$$N(\widetilde{U}) = \langle \widetilde{U}, e_2, e_7, e_{14} \rangle.$$

The subalgebra $\langle e_2, e_7, e_{14} \rangle$ is abelian, and is a torus, whose real part is $\langle e_2, e_7 \rangle$ and compact part is $\langle e_{14} \rangle$.

Thus $N(\widetilde{U})$ is self–normalizing and solvable. Therefore, by Algorithm 3, $\widetilde{U}$ is a maximal ad-nilpotent subalgebra containing U. Using Algorithm 1, we find that $\langle e_2, e_7, e_{14} \rangle$ is a Cartan subalgebra and $A = \langle e_2, e_7 \rangle$ is a maximal abelian subalgebra of real semisimple elements.

The roots of A on $N(\widetilde{U})$ are

$$(-1, 0), (-1, -1), (-1, 1), (0, 1);$$

here, to say that (r, s) is a root means that there is a common eigenvector X for A which is not centralized by A and

$$[e_7, X] = rX, [e_2, X] = sX.$$

Let

$$a = (-1, 0), b = (-1, -1), c = (-1, 1), d = (0, 1).$$

This is a positive system of roots for A determined by $N(\widetilde{U})$. The only positive roots which are sums of positive roots are $a + d = c$ and $b + d = a$. Therefore, the simple roots are b, d and the roots as nonnegative integral combinations of the simple roots are $b, d, b + d, b + 2d$.

Therefore, the Dynkin diagram of the reduced root system is of type B_2 with b a long root.

Let ω_7, ω_2 be linear functions on A dual to the ordered basis e_7, e_2. With this notation, the roots are

$$-\omega_7, -\omega_7 - \omega_2, -\omega_7 + \omega_2, \omega_2.$$

Let $L = N(\widetilde{U})$. The corresponding eigenspaces in L are

$L_{-\omega_7} = \langle e_{12}, e_{11}\rangle, L_{-\omega_7-\omega_2} = \langle e_8+e_{10}\rangle, L_{-\omega_7+\omega_2} = \langle e_8-e_{10}\rangle, L_{\omega_2} = \langle -e_{13}+e_6, -e_{15}+e_4\rangle$.

Finally $L_0 = \langle A, e_{14}\rangle$ and e_{14} operates on these eigenspaces, as rotations on $L_{-\omega_7}$ and L_{ω_2}, while it commutes with $L_{-w_7-w_2}$ and $L_{-\omega_7+\omega_2}$.

The absolute root system is determined by common eigenvectors for the Cartan algebra

$$C = \langle e_7, e_2, e_{14}\rangle .$$

The positive roots are

$$a = (0,1,-\sqrt{-1}), b = (0,1,\sqrt{-1}), c = (1,0,\sqrt{-1})$$
$$d = (1,1,0), e = (1,-1,0), f = (1,0,-1).$$

Forming sums of pairs of positive roots and removing those roots that are sums of positive roots, we find that the simple roots are a, e, b with Dynkin diagram of type A_3 with e the simple middle root.

Conjugation maps a to b and fixes e. Thus the algebra is a real form of $\mathrm{sl}(4,\mathbb{C})$.

To find a maximally compact subalgebra – if any – we follow a procedure analogous to Algorithm 1 – starting with a compact element – namely one which generates a compact subgroup. For example, as $\langle e_{15}, e_{14}, e_{13}\rangle$ generate $\mathfrak{so}(3)$, because $e_{15} = x\partial_y - y\partial_x$ and $e_{14} = x\partial_z - z\partial_x$, we can with begin with e_{15}, compute its centralizer, the center of its centralizer and its derived algebra. If the derived algebra is trivial, then the centralizer e_{15} of would be a maximal torus and its compact part will be a maximal compact subalgebra containing e_{15}. If the derived algebra is nontrivial, it must have a compact element, say t. Adjoining it to e_{15} and computing the centralizer of $\langle e_{15}, t\rangle$ and its derived algebra and repeating the process, we will ultimately obtain a maximal compact subalgebra containing e_{15}.

In this case we find that $C_k = \langle t_1, t_2, e_{15}\rangle$ is a maximally compact Cartan subalgebra, where

$$t_1 = 2e_{12}+e_5, t_2 = e_9+4e_8 .$$

The positive roots are

$$a = (\sqrt{-1}, -4\sqrt{-1}, 0), b = (0, 4\sqrt{-1}, \sqrt{-1}), c = (\sqrt{-1}, 0, -\sqrt{-1}),$$
$$d = (0, 4\sqrt{-1}, -\sqrt{-1}), e = (\sqrt{-1}, 4\sqrt{-1}, 0), f = (\sqrt{-1}, 0, \sqrt{-1}).$$

The simple positive roots are d, a, b with Dynkin diagram of type A_3 with a the middle simple roots.

Conjugation maps every root to its negative. The root algebras generated by the real and imaginary parts of the root vectors are copies of $\mathrm{sl}(2,\mathbb{R})$ except for roots $d+a, a+b$, where they generate copies of $\mathfrak{so}(3)$. Specifically, the subalgebras generated by the real and imaginary parts of root vectors for $d+a$ and $a+b$ are

$$\langle e_1+2e_{10}-2e_{14},\ e_3+2e_{11}+2e_{13},\ -4e_5-8e_{12}+8e_{15}\rangle,$$
$$\langle e_1+2e_{10}+2e_{14},\ -e_3-2e_{11}+2e_{13},\ -4e_5-8e_{12}-8e_{15}\rangle.$$

Both are isomorphic to $\mathfrak{so}(3)$. Denoting these subalgebras by k_1 and k_2 respectively, we find that the centralizer of k_1 is

$$k_2 \oplus \langle 4e_8+e_9\rangle .$$

Moreover as the centralizer of the copy of $\mathfrak{so}(3)$ given by $k_0 = \langle e_{15}, e_{14}, e_{13}\rangle$ is $\langle e_7, e_8, e_9\rangle = \mathrm{sl}(3,\mathbb{R})$, the subalgebra k_1 is not conjugate to k_0.

Finally, as the Killing form of the full algebra has seven negative eigenvalues, a maximal compact subalgebra is

$$k_1 \oplus k_2 \oplus \langle 4e_8+e_9\rangle$$

because $4e_8 + e_9$ generates the maximal compact subalgebra of $\langle e_7, e_8, e_9\rangle = \mathrm{sl}(3,\mathbb{R})$.

5. Lie Symmetries of $f_{xx} = \frac{4}{3}f_{yy}^3, f_{xy} = f_{yy}^2$ and $v' = (u'')^2$

These equations were considered by Cartan in a geometrical context, [Ca], who showed that their symmetry algebra was the 14 dimensional simple group G_2. Maple is able to compute both the algebras by using commands for contact symmetries and for generalized symmetries as well as its root space decomposition and its maximal compact subalgebra. The latter equation was also considered by Anderson, Kamran and Olver, [AKO], in the context of generalized symmetries. To illustrate the algorithms of this paper, we will use the table given in [AKO] – reproduced in Appendix 2 to identify the algebra and determine several interesting subalgebras. To streamline the calculations we will use repeatedly the following facts-already mentioned in the Introduction.

If H is a semisimple subalgebra of a semisimple algebra G then and element X of H is real semisimple, compact or nilpotent in the adjoint representation H of on itself,if and only if it is, respectively, real semisimple, compact or nilpotent in the adjoint representation of H in G. Moreover, the derived algebras of the radical of normalizer or centralizer of any subalgebra are nilpotent.

The symmetry algebra has basis $X_1, X_2, ..., X_{14}$ and is given by:

$$X_1 = (\frac{2}{3}u'^2 - uu'')\partial_x + (\frac{1}{2}uv + \frac{4}{9}u'^3 - uu'u'')\partial_u + (\frac{1}{2}v^2 - \frac{1}{3}uu''^3)\partial_v,$$
$$X_2 = (\frac{4}{3}x^2u' - 2xu - \frac{1}{3}x^3u'')\partial_x + (\frac{1}{6}x^3v + \frac{2}{3}x^2u'^2 - 2u^2 - \frac{1}{3}x^3u'u'')\partial_u + (2xu'v - 2uv - \frac{1}{9}x^3u''^3 - \frac{8}{9}u'^3)\partial_v,$$
$$X_3 = (\frac{8}{3}xu' - 2u - x^2u'')\partial_x + (\frac{1}{2}x^2v + \frac{4}{3}xu'^2 - x^2u'u'')\partial_u + (2vu' - \frac{1}{3}x^2u''^3)\partial_v,$$
$$X_4 = (\frac{8}{3}u' - 2xu'')\partial_x + (xv + \frac{4}{3}u'^2 - 2xu'u'')\partial_u - \frac{2}{3}xu''^3\partial_v,$$

$$X_5 = -2u''\partial_x + (v - 2u'u'')\partial_u - \frac{2}{3}u''^3\partial_v,$$
$$X_6 = \frac{1}{2}u\partial_u + v\partial_v,$$
$$X_7 = -\frac{1}{2}x^2\partial_x - \frac{3}{2}xu\partial_u - 2u'^2\partial_v,$$
$$X_8 = -x\partial_x - \frac{3}{2}u\partial_u,$$
$$X_9 = -\partial_x,$$
$$X_{10} = \frac{1}{6}x^3\partial_u + 2(xu' - u)\partial_v,$$
$$X_{11} = \frac{1}{2}x^2\partial_u + 2u'\partial_v,$$
$$X_{12} = x\partial_u,$$
$$X_{13} = \partial_u,$$
$$X_{14} = \partial_v.$$

Using the commutator table in Appendix 2, and computing the determinant of the Killing form by Maple, we find that it is non-zero. Thus, the algebra is semisimple. The translations

$$X_{14} = \partial_v \; X_{13} = \partial_u$$

clearly commute and $X_{12} = x\partial_u$ commutes with both. One can check that they are nilpotent: this also follows from computing the derived algebra of the radical of normalizer of U. We want to embed

$$U = \langle X_{14}\,, X_{13}\,, X_{12} \rangle$$

in a maximal subalgebra whose elements are all nilpotent. Following Algorithm 3 we compute the normalizer $N(U)$ and find its Levi decomposition, using standard Maple commands:

$$N(U) = \langle X_9\,, X_8\,, X_6\,, X_5\,, X_{14}\,, X_{13}\,, X_{12}\,, X_{11}\,, X_{10} \rangle$$

and its Levi decomposition is $R(N(U)) \oplus S$, where the radical

$$R(N(U)) = \langle X_9\,, X_8 - 3X_6\,, X_{14}\,, X_{13}\,, X_{12}\,, X_{11} \rangle$$

and the semisimple part $S = \langle X_8 + X_6\,, X_5\,, X_{10} \rangle$. The commutators for the semisimple part are

$$[X_8 + X_6\,, X_5] = 2X_5,\; [X_8 + X_6\,, X_{10}] = -2X_{10},\; [X_5,\, X_{10}] = -2(X_8 + X_6)\,.$$

This means that X_5 and X_{10} are nilpotent in the full algebra, $X_8 + X_6$ is real semisimple in the full algebra and $X_5 + X_{10}$ is a compact element. Following Algorithm 3, we compute the derived algebra of the radical $R(N(U))$. It is

$$\widetilde{U} = \langle X_9\,, X_{14}\,, X_{13}\,, X_{12}\,, X_{11} \rangle\,.$$

The quotient $R(N(U))/\widetilde{U}$ is represented by $X_8 - 3X_6$, which is a real semisimple element. (This also follows by computing the centralizer of $X_5 + X_{10}$ and its derived algebra, which turns out to be $\langle X_8 - 3X_6\,, X_{12}\,, X_3 \rangle$. This is a standard $\mathrm{sl}(2, \mathbb{R})$ with $X_8 - 3X_6$ as real semisimple element.)

Following Algorithm 3 we compute again $N(\widetilde{U})$ and its Levi decomposition. It turns out to be identical to the Levi decomposition of $N(U)$. We therefore adjoint a nilpotent element coming from the semisimple part of the decomposition, say X_5. Let

$$\widetilde{\widetilde{U}} = \langle \widetilde{U}, X_5 \rangle\,.$$

Its normalizer is $\langle \widetilde{\widetilde{U}}\,, X_8\,, X_6 \rangle$ and it is solvable, with commutator $\widetilde{\widetilde{U}}$ and the quotient is represented by the real torus $\langle X_6\,, X_8 \rangle$. This also follows from noticing that $X_8 + X_6$ is also real semisimple and commutes with $X_8 - 3X_6$.

Thus a maximal nilpotent subalgebra containing

$$U = \langle X_{14}\,, X_{13}\,, X_{12} \rangle$$

is $\widetilde{\widetilde{U}} = \langle X_5\,, X_{14}\,, X_{13}\,, X_{12}\,, X_{11}\,, X_9 \rangle$. Finally, $C = \langle X_6\,, X_8 \rangle$ is self-centralizing and it is a real split Cartan subalgebra of the full 14 dimensional algebra L.

Maple gives the following roots for C in $\widetilde{\widetilde{U}}$: in fact, the basis vectors for $\widetilde{\widetilde{U}}$ listed above are common eigen-vectors for C with eigenvalues

$$a = (\frac{1}{2}, \frac{3}{2}),\; b = (-1, 0),\; c = (-\frac{1}{2}, \frac{3}{2}),\; d = (-\frac{1}{2}, \frac{1}{2}),\; e = (-\frac{1}{2}, -\frac{1}{2}),\; f = (0, 1)\,.$$

As explained in Section 2, this is a positive system of roots and a simple system of roots is given by adding pairs of positive roots and removing those that are a sum of positive roots. We have

$$a + b = c,\; a + e = f,\; d + e = b,\; e + f = d\,.$$

Thus the simple roots are a, e and the positive roots written in terms of these roots are

$$a,\; e,\; a + e = f,\; a + 2e = d,\; a + 3e = b,\; 2a + 3e = c\,.$$

Therefore, the algebra L is of type G_2 with a real split Cartan subalgebra. Any semisimple split real Lie algebra is generated by copies of $\mathrm{sl}(2,\mathbb{R})$ corresponding to the simple roots, with relations

$$[X\,,Y] \,=\, H,\,[H\,,X] \,=\, 2X,\,[H\,,Y] \,=\, -2Y$$

and its maximal compact subalgebra is generated by copies of the compact element $X-Y$, which generates a circle, in these generating root $\mathrm{sl}(2,\mathbb{R})$ copies; see [St, pp. 99–100] for a global version of these results.

Here, the root vectors corresponding to $a,-a$ are X_5, X_{10}; the root vectors corresponding to $e,-e$ are X_{11}, X_4 and a maximal compact compact subalgebra K is thus generated by

$$J_1 \,=\, X_5+X_{10},\, J_2 \,=\, X_4-X_{11}\,.$$

The algebra is spanned by

$$J_1,\, J_2,\, J_3 \,=\, X_1+\frac{3}{8}X_{14},\, J_4 \,=\, X_2-\frac{3}{4}X_{13},\, J_5 \,=\, X_3+\frac{3}{4}X_{12},\, J_6 \,=\, X_7-\frac{3}{2}X_9\,.$$

To identify the structure of K, we must choose a Cartan subalgebra of K and compute its roots in the complexification of K — exactly as for $\mathfrak{so}(4)$ in Section 4.1. Now the centralizer of J_1 is $\langle J_1\,, J_5\rangle$, and it is therefore a Cartan subalgebra of K. Its positive roots are

$$(\sqrt{-2}\,,-\frac{1}{\sqrt{-2}}),\,(\sqrt{-2}\,,\frac{3}{\sqrt{-2}})\,.$$

Therefore the system is of type $A_1\times A_1$. The real and imaginary parts for the root vectors of $(\sqrt{-2}\,,-\frac{1}{\sqrt{-2}})$ are

$$J_3-\frac{1}{6}J_6,\,\frac{\sqrt{2}}{4}J_4-\frac{\sqrt{2}}{8}J_2 \tag{1}$$

and for the root $(\sqrt{-2}\,,\frac{3}{\sqrt{-2}})$ they are

$$J_3+\frac{1}{2}J_6,\,\frac{-\sqrt{2}}{4}J_4+\frac{3\sqrt{2}}{8}J_2\,. \tag{2}$$

The vectors in (1) generate a copy of $\mathfrak{so}(3)$ and in (2) also a copy of $\mathfrak{so}(3)$ and these subalgebras commute.

This gives an explicit decomposition of a maximal compact subalgebra of L as a sum of copies of $\mathfrak{so}(3)$.

6. Solutions of the wave equation

Section 5 gives several non-cojugate subalgebras of the symmetry algebra of the wave equation. In this section we give explicit invariant solutions using these subalgebras.

The equation is

$$\frac{\partial^2}{\partial x^2}u\,(t,x,y,z)+\frac{\partial^2}{\partial y^2}u\,(t,x,y,z)+\frac{\partial^2}{\partial z^2}u\,(t,x,y,z)-\frac{\partial^2}{\partial t^2}u\,(t,x,y,z)=0 \tag{3}$$

(I) $dimG'=0$

(I.a) $\mathcal{L}_{1,0} =< e_8, e_{10}, e_{11} >$

The joint invariants of $\mathcal{L}_{1,0}$ are

$$z,u$$

so that the corresponding similarity transformations

$$p=z, w(p)=u \tag{4}$$

transform wave equation (3) to

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2} w(p) = 0 \tag{5}$$

which has the solution

$$w(p) = C_1\, p + C_2. \tag{6}$$

This leads to solution

$$u(t,x,y,z) = C_1\, z + C_2 \tag{7}$$

of wave equation (3).

(I.b) $\mathcal{L}_{2,0} =< e_2, e_7 - e_{16}, e_{14} >$

The joint invariants of $\mathcal{L}_{2,0}$ are

$$-\frac{x^2+z^2}{y^2-t^2}, u\sqrt{-y^2+t^2}$$

so that the corresponding similarity transformations

$$p = -\frac{x^2+z^2}{y^2-t^2}, w(p) = u\sqrt{-y^2+t^2} \tag{8}$$

transform wave equation (3) to Jacobi ODE

$$4\left(\frac{\mathrm{d}^2}{\mathrm{d}p^2} w(p)\right)p^2 - 4\left(\frac{\mathrm{d}^2}{\mathrm{d}p^2} w(p)\right)p + 8\left(\frac{\mathrm{d}}{\mathrm{d}p} w(p)\right)p - 4\,\frac{\mathrm{d}}{\mathrm{d}p} w(p) + w(p) = 0 \tag{9}$$

which has the solution

$$w(p) = C_1\, \mathit{EllipticK}\left(\sqrt{p}\right) + C_2\, \mathit{EllipticCK}\left(\sqrt{p}\right) \tag{10}$$

in terms of complete and complementary complete elliptic integrals of the first kind (ref: http://www.maplesoft.com/support/help/Maple/view.aspx?path=EllipticF). This leads to solution

$$u(t,x,y,z) = \frac{1}{\sqrt{-y^2+t^2}}\left(C_1\, \mathit{EllipticK}\left(\sqrt{\frac{-x^2-z^2}{y^2-t^2}}\right) + C_2\, \mathit{EllipticCK}\left(\sqrt{\frac{-x^2-z^2}{y^2-t^2}}\right)\right) \tag{11}$$

of wave equation (3).

(I.c) $\mathcal{L}_{3,0} =< e_{12} + \frac{1}{2}e_5, e_9 + 4e_8, e_{15} >$

The joint invariants of $\mathcal{L}_{3,0}$ are

$$\frac{-t^4 + \left(2\,x^2 + 2\,y^2 + 2\,z^2 - 8\right)t^2 - x^4 + \left(-2\,y^2 - 2\,z^2\right)x^2 - y^4 - 2\,y^2z^2 - \left(z^2+4\right)^2}{4\,x^2 + 4\,y^2}, u\sqrt{x^2+y^2}$$

so that the corresponding similarity transformations

$$p = \frac{-t^4 + \left(2\,x^2 + 2\,y^2 + 2\,z^2 - 8\right)t^2 - x^4 + \left(-2\,y^2 - 2\,z^2\right)x^2 - y^4 - 2\,y^2z^2 - \left(z^2+4\right)^2}{4\,x^2 + 4\,y^2}, \tag{12}$$

$$w(p) = u\sqrt{x^2+y^2} \tag{13}$$

transform wave equation (3) to

$$4\left(\frac{\mathrm{d}^2}{\mathrm{d}p^2} w(p)\right)p^2 + 8\left(\frac{\mathrm{d}}{\mathrm{d}p} w(p)\right)p - 16\,\frac{\mathrm{d}^2}{\mathrm{d}p^2} w(p) + w(p) = 0 \tag{14}$$

which has the solution

$$w(p) = C_1\, \mathit{LegendreP}\left(-1/2, p/2\right) + C_2\, \mathit{LegendreQ}\left(-1/2, p/2\right) \tag{15}$$

in terms of Legendre functions of the first and second kind
(ref: http://www.maplesoft.com/support/help/Maple/view.aspx?path=Legendre). This leads to solution

$$u(t,x,y,z) = \frac{1}{\sqrt{x^2+y^2}}\left(C_1\ LegendreP\left(-1/2, p/2\right) + C_2\ LegendreQ\left(-1/2, p/2\right)\right) \tag{16}$$

of wave equation (3) where p is given by (12).

(II) $dimG' = 1$

(II.a) $\mathcal{L}_{1,1} =< e_2, e_7 - e_{16}, e_8 + e_{10} >$

The joint invariants of $\mathcal{L}_{1,1}$ are

$$\frac{z}{x}, ux$$

so that the corresponding similarity transformations

$$p = \frac{z}{x}, w(p) = ux \tag{17}$$

transform wave equation (3) to

$$\left(\frac{d^2}{dp^2}w(p)\right)p^2 + 4\left(\frac{d}{dp}w(p)\right)p + 2\,w(p) + \frac{d^2}{dp^2}w(p) = 0 \tag{18}$$

which has the solution

$$w(p) = \frac{C_1\,p + C_2}{p^2+1}. \tag{19}$$

This leads to solution

$$u(t,x,y,z) = \frac{C_1\,z + C_2\,x}{x^2+z^2} \tag{20}$$

of wave equation (3).

(II.b) $\mathcal{L}_{2,1} =< e_{12}, -e_6 + e_{13}, -e_8 + e_{10} >$

The joint invariants of $\mathcal{L}_{2,1}$ are

$$x, t+y, u$$

which gives the similarity transformation

$$p = x, q = t + y, w(p,q) = u \tag{21}$$

that transforms the wave equation into

$$w_{pp} = 0, \tag{22}$$

which gives the solution

$$u(t,x,y,z) = xF_1(y+t) + F_2(y+t). \tag{23}$$

(III) $dimG' = 2$

(III.a) $\mathcal{L}_{1,2} =< e_7 - e_{16}, e_{11}, e_{12} >$

The joint invariants of $\mathcal{L}_{1,2}$ are

$$\frac{y}{t}, ut$$

so that the corresponding similarity transformations

$$p = \frac{y}{t}, w(p) = ut \tag{24}$$

transform wave equation (3) to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}p^2}w\left(p\right)\right)p^2+4\left(\frac{\mathrm{d}}{\mathrm{d}p}w\left(p\right)\right)p+2\,w\left(p\right)-\frac{\mathrm{d}^2}{\mathrm{d}p^2}w\left(p\right)=0 \tag{25}$$

which has the solution

$$w\left(p\right)=\frac{C_1\,p+C_2}{p^2-1}. \tag{26}$$

This leads to solution

$$u\left(t,x,y,z\right)=\frac{C_1\,y+C_2\,t}{y^2-t^2} \tag{27}$$

of wave equation (3).

(III.b) $\mathcal{L}_{2,2}=<e_2,-e_6+e_{13},-e_4+e_{15}>$

The joint invariants of $\mathcal{L}_{2,2}$ are

$$x^2+y^2+z^2-t^2,u$$

so that the corresponding similarity transformations

$$p=x^2+y^2+z^2-t^2,w(p)=u \tag{28}$$

transform wave equation (3) to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}p^2}w\left(p\right)\right)p+2\,\frac{\mathrm{d}}{\mathrm{d}p}w\left(p\right)=0 \tag{29}$$

which has the solution

$$w\left(p\right)=\frac{C_1\,p+C_2}{p}. \tag{30}$$

This leads to solution

$$u\left(t,x,y,z\right)=C_1+\frac{C_2}{x^2+y^2+z^2-t^2} \tag{31}$$

of wave equation (3).

(III.c) $\mathcal{L}_{3,2}=<e_{14},e_{11},e_{12}>$

The joint invariants of $\mathcal{L}_{3,2}$ are

$$t,y,u$$

so that the corresponding similarity transformations

$$p=t,q=y,w(p,q)=u \tag{32}$$

transform wave equation (3) to

$$\frac{\partial^2}{\partial q^2}w\left(p,q\right)-\frac{\partial^2}{\partial p^2}w\left(p,q\right)=0 \tag{33}$$

which has the solution

$$w\left(p,q\right)=F_1\left(q+p\right)+F_2\left(q-p\right). \tag{34}$$

This leads to solution

$$u\left(t,x,y,z\right)=F_1\left(y+t\right)+F_2\left(y-t\right) \tag{35}$$

of wave equation (3).

(III.d) $\mathcal{L}_{4,2}=<e_{14},-e_6+e_{13},-e_4+e_{15}>$

The joint invariants of $\mathcal{L}_{4,2}$ are

$$y+t,x^2-2\,t\,y+z^2-2\,t^2,u$$

so that the corresponding similarity transformations

$$p = y + t, q = x^2 - 2\,t\,y + z^2 - 2\,t^2, w(p, q) = u \tag{36}$$

transform wave equation (3) to

$$\left(-p^2 + q\right) \frac{\partial^2}{\partial q^2} w\left(p, q\right) + \left(\frac{\partial^2}{\partial q \partial p} w\left(p, q\right)\right) p + 2\,\frac{\partial}{\partial q} w\left(p, q\right) = 0 \tag{37}$$

which has the solution

$$w\left(p, q\right) = \frac{1}{p}\left(F_2\left(p\right) p + F_1\left(\frac{p^2 + q}{p}\right)\right). \tag{38}$$

This leads to solution

$$u\left(t, x, y, z\right) = F_2\left(y + t\right) + \frac{1}{y + t} F_1\left(\frac{x^2 + y^2 + z^2 - t^2}{y + t}\right) \tag{39}$$

of wave equation (3).

(IV) $dimG' = 3$

(IV.a) $\mathcal{L}_{1,3} =< e_{15}, e_{14}, e_{13} >$

The joint invariants of $\mathcal{L}_{1,3}$ are

$$t, x^2 + y^2 + z^2, u$$

so that the corresponding similarity transformations

$$p = t, q = x^2 + y^2 + z^2, w(p, q) = u \tag{40}$$

transform wave equation (3) to

$$4\left(\frac{\partial^2}{\partial q^2} w\left(p, q\right)\right) q + 6\,\frac{\partial}{\partial q} w\left(p, q\right) - \frac{\partial^2}{\partial p^2} w\left(p, q\right) = 0 \tag{41}$$

which has the solution

$$w\left(p, q\right) = \frac{F_1\left(\sqrt{q} + p\right) + F_2\left(-\sqrt{q} + p\right)}{\sqrt{q}}. \tag{42}$$

This leads to solution

$$u\left(t, x, y, z\right) = \frac{F_1\left(\sqrt{x^2 + y^2 + z^2} + t\right) + F_2\left(-\sqrt{x^2 + y^2 + z^2} + t\right)}{\sqrt{x^2 + y^2 + z^2}} \tag{43}$$

of wave equation (3).

(IV.b) $\mathcal{L}_{2,3} =< e_7 - e_{16}, e_8, e_9 >$

The joint invariants of $\mathcal{L}_{2,3}$ are

$$\frac{y}{x}, \frac{z}{x}, ux$$

so that the corresponding similarity transformations

$$p = \frac{y}{x}, q = \frac{z}{x}, w(p, q) = ux \tag{44}$$

transform wave equation (3) to

$$\left(\frac{\partial^2}{\partial p^2} w\right) p^2 + 2\,pq\frac{\partial^2}{\partial q \partial p} w + \left(\frac{\partial^2}{\partial q^2} w\right) q^2 + 4\,p\frac{\partial}{\partial p} w + 4\left(\frac{\partial}{\partial q} w\right) q + \frac{\partial^2}{\partial p^2} w + 2\,w + \frac{\partial^2}{\partial q^2} w = 0 \tag{45}$$

which has the solution

$$w\left(p, q\right) = \frac{C_1\,p + C_2}{p^2 + 1} + \frac{C_3\,q + C_4}{q^2 + 1}. \tag{46}$$

This leads to solution

$$u\left(t,x,y,z\right)=\frac{1}{x}\left(\frac{\left(C_1\,y+C_2\,x\right)x}{x^2+y^2}+\frac{\left(C_3\,z+C_4\,x\right)x}{x^2+z^2}\right) \tag{47}$$

of wave equation (3).

(IV.c) $\mathcal{L}_{3,3} =< e_1 + 2e_{10} + 2e_{14}, -e_3 - 2e_{11} + 2e_{13}, -4e_5 - 8e_{12} - 8e_{15} >$

The joint invariants of $\mathcal{L}_{3,3}$ are

$$\frac{x^2+y^2+z^2-t^2+4}{t}, ut$$

so that the corresponding similarity transformations

$$p=\frac{x^2+y^2+z^2-t^2+4}{t}, w(p)=ut \tag{48}$$

transform wave equation (3) to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}p^2}w\left(p\right)\right)p^2+4\left(\frac{\mathrm{d}}{\mathrm{d}p}w\left(p\right)\right)p+2\,w\left(p\right)+16\,\frac{\mathrm{d}^2}{\mathrm{d}p^2}w\left(p\right)=0 \tag{49}$$

which has the solution

$$w\left(p\right)=\frac{C_1\,p+C_2}{p^2+16} \tag{50}$$

This leads to solution

$$u\left(t,x,y,z\right)=\frac{-C_1\,t^2+C_2\,t+C_1\,\left(x^2+y^2+z^2+4\right)}{t^4+\left(-2\,x^2-2\,y^2-2\,z^2+8\right)t^2+\left(x^2+y^2+z^2+4\right)^2} \tag{51}$$

of wave equation (3).

Appendices

Appendix 1

Table 1. Commutator table for symmetry algebra of wave equation on Minkowski spacetime

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}
X_1	0	$\frac{-1}{2}X_9$	0	0	0	0	$-X_1$	$-X_2$	0	$X_{16}-X_7$	X_{15}	X_{13}	$-X_5$	0	$-X_3$	0
X_2		0	0	$-X_{15}$	0	$-X_{13}$	0	$-X_{10}$	$2X_1$	$-X_8$	0	0	$-X_6$	0	$-X_4$	0
X_3			0	$\frac{-1}{2}X_9$	0	0	$-X_3$	$-X_4$	0	$-X_{15}$	$X_{16}-X_7$	X_{14}	0	$-X_5$	X_1	0
X_4				0	0	$-X_{14}$	0	$-X_{11}$	$2X_3$	0	$-X_8$	0	0	$-X_6$	X_2	0
X_5					0	$\frac{-1}{2}X_9$	$-X_5$	$-X_6$	0	$-X_{13}$	$-X_{14}$	$X_{16}-X_7$	X_1	X_3	0	0
X_6						0	0	$-X_{12}$	$2X_5$	0	0	$-X_8$	X_2	X_4	0	0
X_7							0	$-X_8$	X_9	$-X_{10}$	$-X_{11}$	$-X_{12}$	0	0	0	0
X_8								0	$2X_7-2X_{16}$	0	0	0	0	0	0	0
X_9									0	$-2X_2$	$-2X_4$	$-2X_6$	0	0	0	0
X_{10}										0	0	0	$-X_{12}$	0	$-X_{11}$	0
X_{11}											0	0	0	$-X_{12}$	X_{10}	0
X_{12}												0	X_{10}	X_{11}	0	0
X_{13}													0	X_{15}	$-X_{14}$	0
X_{14}														0	X_{13}	0
X_{15}															0	0
X_{16}																0

where $[X_i, X_j] = -[X_j, X_i]$.

Appendix 2

Table 2. Commutator table for $G2$

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}
X_1	0	0	0	0	0	$-X_1$	0	0	0	$-\frac{1}{2}X_2$	$-\frac{1}{2}X_3$	$-\frac{1}{2}X_4$	$-\frac{1}{2}X_5$	$-X_6$
X_2		0	0	0	$4X_1$	$-\frac{1}{2}X_2$	0	$\frac{3}{2}X_2$	$-X_3$	0	0	$-\frac{4}{3}X_7$	$-2X_8+2X_6$	$-X_{10}$
X_3			0	$-4X_1$	0	$-\frac{1}{2}X_3$	$-\frac{3}{2}X_2$	$\frac{1}{2}X_3$	$-X_4$	0	$\frac{4}{3}X_7$	$\frac{2}{3}X_8-2X_6$	$2X_9$	$-X_{11}$
X_4				0	0	$-\frac{1}{2}X_4$	$-2X_3$	$-\frac{1}{2}X_4$	$-X_5$	$-\frac{4}{3}X_7$	$\frac{2}{3}X_8+2X_6$	$-\frac{8}{3}X_9$	$-X_{12}$	0
X_5					0	$-\frac{1}{2}X_5$	$-\frac{3}{2}X_4$	$-\frac{3}{2}X_5$	0	$-2X_8-2X_6$	$2X_9$	0	0	$-X_{13}$
X_6						0	0	0	0	$-\frac{1}{2}X_{10}$	$-\frac{1}{2}X_{11}$	$-\frac{1}{2}X_{12}$	$-\frac{1}{2}X_{13}$	$-X_{14}$
X_7							0	X_7	$-X_8$	0	$\frac{3}{2}X_{10}$	$2X_{11}$	$\frac{3}{2}X_{12}$	0
X_8								0	X_9	$-\frac{3}{2}X_{10}$	$-\frac{1}{2}X_{11}$	$\frac{1}{2}X_{12}$	$\frac{3}{2}X_{13}$	0
X_9									0	X_{11}	X_{12}	X_{13}	0	0
X_{10}										0	0	0	$2X_{14}$	0
X_{11}											0	$-2X_{14}$	0	0
X_{12}												0	0	0
X_{13}													0	0
X_{14}														0

where $[X_i, X_j] = -[X_j, X_i]$.

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