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A. Laradji

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: c-math@kfupm.edu.sa

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A. Laradji
Department of Mathematics & Statistics
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia
alaradji@kfupm.edu.sa

Abstract

We provide necessary and sufficient conditions for which an n^{th}-order linear differential equation has n linearly independent polynomial solutions. Necessary conditions that can directly be ascertained from the coefficient functions are also given.[1]

1 Introduction

Consider the equation

$$p_0 y + p_1 y' + \cdots + p_{n-1} y^{(n-1)} + p_n y^{(n)} = 0 \tag{1}$$

where the p_k are functions (of a single variable x) continuous on some real interval over which p_n is nonzero. If this equation has n linearly independent polynomial solutions, then it is easy to see, using Cramer's rule, that each $\frac{p_k}{p_n}$ is a rational function. We will therefore assume, without loss of generality, that the coefficients p_k in (1) are polynomials with no common factor and that p_n is monic.

Determining conditions for which (1) has a fundamental set of polynomial solutions is a problem that has been discussed by several authors when $n = 2$. In [3] for example, Calogero provided conditions for a wide class of second-order linear differential equations to have a general polynomial solution. See also Calogero [4] in connection with a certain class of solvable N-body problems, Calogero and Yi [5] concerning Jacobi polynomials, and Bagchi, Grandati and Quesne [2] on the trigonometric Darboux-Pöschl-Teller potential.

[1] *2010 Mathematics Subject Classification:* 34A05

The main objective in this note is to give conditions, which do not seem to be widely known, for which (1) has a fundamental set of polynomial solutions. In Proposition 1, we provide necessary conditions that can quickly be ascertained from the leading coefficients and degrees of the polynomials p_k. Proposition 2, while computationally more demanding, provides a necessary and sufficient condition.

Let K be the smallest integer k for which p_k in (1) is not the zero polynomial. Clearly, (1) has n linearly independent polynomial solutions if and only if the equation

$$p_K y + p_{K+1} y' + \cdots + p_{n-1} y^{(n-K-1)} + p_n y^{(n-K)} = 0$$

has $n-K$ linearly independent polynomial solutions. We will therefore assume that $p_0 \neq 0$ in (1). For notational convenience, we will write each p_h in (1) in "exponential" form, $p_h = \sum_{k \geq 0} \frac{p_{hk}}{k!} x^k$ with $p_h = 0$ if $h > n$ and, if $p_h \neq 0$, we denote its leading coefficient by γ_h (with $\gamma_n = 1$).

2 Results

We will need the following lemma.

Lemma 1. *Let $r_1 < \cdots < r_n$ be a sequence of nonnegative integers and $y_1, \ldots, y_n$ be monic polynomials with respective degrees $d_1 < \cdots < d_n$. Consider the generalized Wronskian $W\begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix}$, i.e. the determinant of the $n \times n$ matrix whose $(i,j)^{th}$-element is $y_i^{(r_j)}$. Then, either $W\begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix}$ is the zero polynomial or it has degree $\sum_{i=1}^n (d_i - r_i)$ and positive leading coefficient $\det\left(\begin{pmatrix} d_i \\ r_j \end{pmatrix}\right)_{1 \leq i,j \leq n}$. Furthermore, if $W\begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix} = 0$, then $W\begin{pmatrix} y_1, \ldots, y_n \\ s_1, \ldots, s_n \end{pmatrix} = 0$ for any sequence $s_1 < \cdots < s_n$ of nonnegative integers satisfying $r_i \leq s_i$ for all i.*

Proof. Clearly, the degree of $W\begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix}$ does not exceed the sum $\sum_{i=1}^n (d_i - r_i)$ of the degrees of the diagonal polynomials in $W\begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix}$, and the coefficient c of $x^{\sum_{i=1}^n (d_i - r_i)}$ is the same as that of $W\begin{pmatrix} x^{d_1}, \ldots, x^{d_n} \\ r_1, \ldots, r_n \end{pmatrix}$ (since the remaining terms of the y_i have lower order). Applying elementary operations on this determinant, we obtain $c = \det\left(\begin{pmatrix} d_i \\ r_j \end{pmatrix}\right)_{1 \leq i,j \leq n}$. By [6], c is nonnegative and $c > 0$ iff

$r_i \leq d_i$ for each i. This implies $\deg W \begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix} = \sum_{i=1}^{n} (d_i - r_i)$ iff $r_i \leq d_i$ for each i. Suppose $r_i > d_i$ for some i. Then, in the determinant $W \begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix}$, the only possibly nonzero entries in the first i rows are the first $(i-1)$ entries, and so these i rows are linearly dependent and $W \begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix} = 0$. Suppose now that $s_1 < \cdots < s_n$ is a sequence of nonnegative integers satisfying $r_i \leq s_i$ for all i. If $W \begin{pmatrix} y_1, \ldots, y_n \\ s_1, \ldots, s_n \end{pmatrix} \neq 0$, then its leading coefficient is $\det \left(\binom{d_i}{s_j} \right)_{1 \leq i,j \leq n}$, which is positive by the foregoing argument. Hence $d_i \geq s_i \geq r_i$ for all i, and $\det \left(\binom{d_i}{r_j} \right)_{1 \leq i,j \leq n} > 0$. This implies $W \begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix} \neq 0$.

Proposition 2. *Suppose (1) has n linearly independent polynomial solutions. Then, for $0 \leq k \leq n$,*

(i) $p_k \neq 0$

(ii) $1 + \deg p_k = \deg p_{k+1}$

(iii) γ_k *and* γ_{k+1} *are integers with opposite signs.*

(iv) $\deg p_n \leq -\gamma_{n-1}$.

Proof. Let $y_1, \ldots, y_n$ be linearly independent polynomial solutions of the differential equation. We may clearly assume that the y_i are monic with respective degrees d_i where $d_1 < \cdots < d_n$. Let $r_{k1}, \ldots, r_{kn}$ denote the strictly increasing sequence consisting of the elements of $\{0, 1, 2, \ldots, n\} \backslash \{k\}$ and let $W_k := W \begin{pmatrix} y_1, \ldots, y_n \\ r_{k1}, \ldots, r_{kn} \end{pmatrix}$ be the $n \times n$ determinant defined in the above lemma. A straightforward application of Cramer's rule to the system $\frac{p_0}{p_n} y_i + \cdots + \frac{p_{n-1}}{p_n} y_i^{(n-1)} = -y_i^{(n)}$ $(1 \leq i \leq n)$, with appropriate interchanges of columns, shows that for $0 \leq k \leq n$

$$p_k W_n = (-1)^{n-k} p_n W_k \tag{2}$$

By the above lemma, if $W_k \neq 0$, then its leading coefficient $c_k = \det \left(\binom{d_i}{r_{kj}} \right)$ $(1 \leq i, j \leq n)$ is positive. Clearly, in this case,

$$\deg W_k = \sum_{i=1}^{n} (d_i - r_{ki}) = \sum_{i=1}^{n} d_i + k - \frac{n(n+1)}{2}$$

If $p_k \neq 0$, then

$$\deg p_k + \deg W_n = \deg p_n + \sum_{i=1}^{n} d_i + k - \frac{n(n+1)}{2}$$

and in particular,

$$\deg p_k - k = \deg p_n - n \tag{3}$$

This shows that if p_k and p_{k+1} are nonzero polynomials, then $\deg p_{k+1} = 1 + \deg p_k$. Suppose next that $k < n$ and $p_{k+1} = 0$. Clearly, $r_{k+1,j} \leq r_{kj}$ for each j. Hence, by the lemma, $W_k = 0$ when $W_{k+1} = 0$, i.e. $p_k = 0$. This proves (i) and (ii).
To prove (iii), fix i in $\{1, \ldots n\}$. For $0 \leq k \leq n$, if $p_k y_i^{(k)}$ is a nonzero polynomial, then it has degree $\deg p_k - k + d_i$ and leading coefficient $\gamma_k (d_i)_k$, where, for any number a, $(a)_k := a(a-1)\cdots(a-k+1)$, $(a)_0 := 1$. Hence, by Eq. (3), we obtain for each i,

$$\sum_{k=0}^{n} \gamma_k (d_i)_k = 0$$

i.e. the polynomial

$$f(x) = \sum_{k=0}^{n} \gamma_k (x)_k$$

which has degree n, has the n distinct nonnegative integer roots $d_1, \ldots, d_n$. It is then easy to infer that each γ_k is an integer. Part (iii) of the proposition now follows from Equation (2).
Suppose next that $p_n = \prod_j (x - r_j)$ where the r_j are the (not necessarily distinct) complex roots of p_n. By Eq. (2), each r_j is a root of W_n (otherwise it would be a common root for all the p_k), and so p_n divides W_n. Also, the coefficient $-\sum_{i=1}^{n} d_i$ of x^{n-1} in the polynomial $f(x)$ can easily shown to be $\gamma_{n-1} - \dfrac{n(n-1)}{2}$, so that $-\gamma_{n-1} = \sum_{i=1}^{n} d_i - \dfrac{n(n-1)}{2} = \deg W_n \geq \deg p_n$. This proves (iv). ■

Remark 3. Suppose (1) has a fundamental set of polynomial solutions. Then, using the above notation:
1. The strictly increasing sequence of degrees d_i of the polynomial solutions of (1) satisfy $\prod_{i=1}^{n} d_i = (-1)^n \gamma_0$ and hence $n! \leq |\gamma_0|$. Similar bounds involving other leading coefficients γ_k can easily be obtained using the fact that $\prod_{i=1}^{n} (x - d_i) = \sum_{k=0}^{n} \gamma_k (x)_k$. Several upper bounds can also be obtained for d_n, the highest degree possible for a polynomial solution of (1); for example $d_n \leq \dfrac{|\gamma_0|}{(n-1)!}$. Note also that if $|\gamma_0|$ is a prime power p^α, then the degrees of all polynomial solutions of (1) are also powers of p and $\dfrac{n(n-1)}{2} \leq \alpha$ (by a simple argument on partitions of α into distinct nonnegative parts), i.e. $n \leq \dfrac{\sqrt{1+8\alpha}-1}{2}$.

2. By Equation (2), $\frac{p_{n-1}}{p_n} = -\frac{W_{n-1}}{W_n}$. It is easy to see that $\frac{dW_n}{dx} = W_{n-1}$, so that $e^{-\int \frac{p_{n-1}}{p_n} dx}$ is a polynomial of the same degree as W_n, i.e. its degree is $-\gamma_{n-1}$. This also means that if $p_n = \prod_j (x - r_j)$, then $\frac{p_{n-1}}{p_n} = \sum_j \frac{A_j}{x - r_j}$ where the A_j are negative integers. In particular, all the roots of p_n that are not roots of p_{n-1} must be simple.

We next give a necessary and sufficient condition for (1) to have polynomial solutions only. By the above proposition, we will assume that $\deg p_h = h + d$ $(0 \le h \le n)$ where $d := \deg p_0$, so that $p_{hk} = 0$ if $k > h + d$. Recall that p_h has leading coefficient γ_h and that $p_h = \sum_{k \ge 0} \frac{p_{hk}}{k!} x^k$ with $p_h = 0$ if $h > n$.

Proposition 4. *For each positive integer N, let A_N be the $(N + d + 1) \times (N + 1)$ matrix with $(i, j)^{\text{th}}$ entry $\sum_{k=0}^{i-1} \binom{i-1}{k} p_{k+j-i,k}$. Then (1) has a fundamental set of polynomial solutions if and only if there is a strictly increasing sequence of positive integers $d_1, \ldots, d_n$ such that $\text{rank} A_{d_t} = \text{rank} A'_{d_t}$ $(1 \le t \le n)$, where A'_{d_t} is the matrix obtained from A_{d_t} by deleting its last column. In this case, $d_1, \ldots, d_n$ are the roots of the polynomial $\sum_{k=0}^{n} \gamma_k (x)_k$ (and are the degrees of n linearly independent solutions of (1)).*

Proof. Suppose that (1) has a polynomial solution of degree N, $q = \sum_{l \ge 0} \frac{q_l}{l!} x^l$ say (with $q_l = 0$ if $l > N$). Direct substitution gives $\sum_{h \ge 0} p_h \sum_{l \ge 0} \frac{q_{h+l}}{l!} x^l = 0$, i.e.

$$\sum_{h \ge 0} \sum_{k \ge 0} \sum_{l \ge 0} \binom{k+l}{k} p_{hk} q_{h+l} \frac{x^{k+l}}{(k+l)!} = 0.$$

Relabeling indices gives

$$\sum_{u \ge 0} \sum_{h \ge 0} \sum_{0 \le k \le u} \binom{u}{k} p_{hk} q_{u+h-k} \frac{x^u}{u!} = 0 \quad (0 \le u \le N + d).$$

We therefore have $\sum_{h \ge 0} \sum_{0 \le k \le u} \binom{u}{k} p_{hk} q_{u+h-k} = 0$, which gives the following system in the unknowns q_l (with the convention that $p_{hk} = 0$ if $h < 0$)

$$\sum_{l=0}^{N} \sum_{0 \le k \le u} \binom{u}{k} p_{l+k-u,k} q_l = 0 \quad (0 \le u \le N + d) \,. \tag{4}$$

Let A_N be as in the statement of the proposition, i.e. the $(N+d+1)\times(N+1)$ coefficient matrix $\left[\sum_{k=0}^{i-1}\binom{i-1}{k}p_{k+j-i,k}\right]$ of the system. It is then easy to show that this system has a solution $(q_0,\ldots,q_N)$ with $q_N\neq 0$ if and only if the rank of A_N is equal to the rank of the matrix A'_N obtained from A_N by deleting its last column (see [1, Lemma 1]). If, conversely, the system (4) has a solution $(q_0,\ldots,q_N)$ with $q_N\neq 0$, then, clearly, (1) will have a polynomial solution of degree N. ■

Remark 5. The polynomial solutions of (1) can easily be computed from the matrices A_N: if we denote by $C_0,\ldots,C_N$ the columns of A_N, then, from the proof above, the coefficients of a monic solution $q_0+q_1x+\cdots+q_{N-1}x^{N-1}+x^N$ of (1) satisfy $C_N=-q_0C_1-\cdots-q_{N-1}C_{N-1}$.

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