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EQUALITY OF THE ALGEBRAIC AND GEOMETRIC RANKS OF CARTAN SUBALGEBRAS AND APPLICATIONS TO LINEARIZATION OF A SYSTEM OF RDINARYDIFFERENTIAL EQUATIONS

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Abstract. If L is a semisimple Lie algebra of vector fields on $\mathbb{R}^N$ with a split Cartan subalgebra C, then it is proved that the dimension of the generic orbit of C coincides with the dimension of C. As a consequence one obtains a local canonical form of L in terms of exponentials of coordinate functions and vector fields that are independent of these coordinates – for a suitable choice of coordinates. This result is used to classify semisimple algebras of vector fields on $\mathbb{R}^3$ and to determine all representations of $\mathrm{sl}(N, \mathbb{R})$ as vector fields on $\mathbb{R}^N$. These representations are used to find linearizing coordinates for any second order ordinary differential equation that admits $\mathrm{sl}(3, \mathbb{R})$ as its symmetry algebra and for a system of two second order ordinary differential equations that admits $\mathrm{sl}(4, \mathbb{R})$ as its symmetry algebra.

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1. Introduction

Our aim here is to illustrate a general observation – namely, that the presence of large abelian subalgebras of the symmetry algebra of differential equations – or systems thereof – is behind the various schemes for finding coordinates that linearize a given linearizable equation.

This is illustrated in detail for a single ordinary differential equation and a system of two ordinary differential equations. This is achieved by investigating the local canonical form of semisimple algebras of vector fields on $\mathbb{R}^N$.

As is well known, the symmetry algebra of the equation $y''(x) = 0$ is $\mathrm{sl}(3, \mathbb{R})$ and the symmetry algebra of the system of two equations $y''(x) = 0$, $z''(x) = 0$ is $\mathrm{sl}(4, \mathbb{R})$; see [Li1], [Ib]. More generally, the symmetry algebra of a system of n free particles is $\mathrm{sl}(n + 2, \mathbb{R})$ [GL], [AKM].

Using the local canonical forms of semisimple algebras of vector field, the representations of $\mathrm{sl}(n, \mathbb{R})$ as vector fields on $\mathbb{R}^{n-1}$ are determined. Declaring one variable as the independent and the remaining as dependent variables and computing the invariants in the second prolongations gives the canonical forms of the equations considered as well as the linearizing coordinates coming from abelian subalgebras of maximal dimension and maximal rank.

The methods used give at the same time a list, in terms of root systems, of semisimple subalgebras of vector fields on $\mathbb{R}^3$. The classification of all subalgebras of vector fields on $\mathbb{R}^3$ is one of the main topics in Ch. 1–4 of [Li2]; the reader is referred to the survey [Ko].

We have organized this paper in two parts. Section 2 deal with general properties of semisimple Lie algebras of vector fields, and the rest deals with applications to linearization.

2. Geometric rank of Cartan subalgebras

The geometric rank of a Lie algebra of vector fields is the maximum of the dimensions of the leaves of the corresponding foliation. It is same as the dimension of the generic leaf. The main result of this section is as follows:

Theorem 2.1. *Let L be a semisimple Lie algebra of vector fields on $\mathbb{R}^N$ which has a split Cartan subalgebra C. Then the dimension of C equals the geometric rank of C.*

Moreover, in suitable coordinates $x_1, x_2, \cdots, x_N$, the root spaces corresponding to a given system of simple roots and their negatives are of the form $\exp(x_i)V_i$, $\exp(-x_i)W_i$, $i = 1, \cdots, n$, where V_i and W_i are vector fields whose coefficients with respect to the basis $\partial_{x_1}, \cdots, \partial_{x_N}$ are independent of $x_1, \cdots, x_n$, and the linear span of the system of vector fields

$$[\exp(x_i)V_i, \exp(-x_i)W_i] = H_i\,, \quad i = 1, \cdots, n$$

is that of $\{\partial_{x_1}, \cdots, \partial_{x_n}\}$.

For any $1 \leq i \leq n$, the Lie algebra generated by $\exp(x_i)V_i$, $\exp(-x_i)W_i$ is a copy of $\mathrm{sl}(2, \mathbb{R})$.

Theorem 2.1 is very useful in finding – up to local equivalence – semisimple algebras of vector fields in $\mathbb{R}^N$ for $N \le 3$. The local classification of all Lie algebras of vector fields in $\mathbb{R}^N$, $N \le 3$, is one of the main themes in S. Lie's work [Li2].

The importance of such a classification is in the canonical forms of differential equations for which Lie invented his theory of prolongations and differential invariants.

For an abelian algebra, the geometric rank and dimension, in general, are not equal. For example the linearly independent vector fields ∂_x, $y\partial_x$, $y^2\partial_x$, $\cdots$, $y^k\partial_x$ commute for every k. Also, for any Lie algebra of vector fields on $\mathbb{R}^N$ the dimension of its generic orbit is the same as the geometric rank of a certain abelian algebra of vector fields – as shown in [ABGM].

Recall that if L is a Lie algebra of vector fields on $\mathbb{R}^N$ then the generic rank of the matrix of coefficients of a basis of L in the standard basis $\partial_{x_1}, \cdots , \partial_{x_N}$ of vector fields on $\mathbb{R}^N$ is the dimension of a generic leaf for L. It is thus an invariant of the algebra L.

2.1. **Preliminaries and notation.** The proof of Theorem 2.1 uses the following standard results about roots of a Lie algebra, for which the reader is referred to [Bo], [HN], [Kn]

Let C be a split Cartan subalgebra of L and R its root system. Let $S = \{\alpha_1, \cdots , \alpha_n\}$ be the simple roots of C for a choice of positive roots. The root spaces are subspaces of L normalized but not centralized by C. The corresponding linear functions are the roots of C. Each root space is one dimensional and a nonzero vector in a root space is called a root vector. For each root r we choose a nonzero element X_r in the corresponding root space. The algebra L is generated as a vector space by the Cartan subalgebra C and the root vectors $\{X_r\}_{r\in R}$ with

$$[X_r,\, X_s] \,=\, N_{r,s}X_{r+s}\,,$$

where $N_{r,s} \neq 0$ if and only if $r + s$ is a root. Moreover $[X_r, X_{-r}] = H_r$ is a nonzero element of C and the Lie algebra generated by the pair of root vectors $\{X_r, X_{-r}\}$ is a copy of $\mathrm{sl}(2,\mathbb{R})$.

The root vectors X_{α_i}, $X_{-\alpha_i}$, $i = 1, \cdots , n$, generate L as a Lie algebra. For a simple root α_i, we denote the element $[X_{\alpha_i}, X_{-\alpha_i}]$ by H_i.

2.2. **Proof of Theorem 2.1.** Let n be the dimension of the Cartan subalgebra C. If $n = 1$ then – using the notation of Section 2.1 – in a neighborhood of a point where H_1 is not zero, we can find coordinates $x_1, \cdots , x_N$ in which $H_1 = \partial_{x_1}$. The vector fields V which are eigenvectors of H_1 in the sense that $[H_1, V] = \lambda \cdot V$ are of the form $V = \exp(\lambda x_1)U$, where U is a vector field whose coefficients in the basis $\partial_{x_1}, \cdots , \partial_{x_N}$ are independent of x_1. Our algebra is generated by eigenvectors of H_1 for nonzero and opposite eigenvalues. Thus if we substitute λx_1 in place of x_1, and leave the other coordinates unchanged, then in this coordinate system the algebra is generated by vector fields $\exp(x_1)V_1$, $\exp(-x_1)W_1$, where V_1 and W_1 are vector fields whose coefficients in the basis $\partial_{x_1}, \cdots , \partial_{x_N}$ are independent of x_1.

We will employ induction. Assume that the theorem is proved for all C with $\dim C \le m$.

Now let the dimension of the Cartan algebra C be $n = m + 1$.

The algebra generated by the root vectors $\{X_{\alpha_i},\, X_{-\alpha_i}\}_{i=1}^m$ is semisimple. By the induction hypothesis, the rank and dimension of the system of vector fields $\{H_i\}_{i=1}^m$ is m. As these are commuting vector fields, we can introduce coordinates in which $H_i = \partial_{x_i}$, $1 \leq i \leq m$. Moreover, again by the induction hypothesis, root vectors of this subalgebra corresponding to the simple roots and their negatives are of the form $\exp(x_i)V_i$, $\exp(-x_i)W_i$, $1 \leq i \leq m$, where V_i and W_i are vector fields whose coefficients in the basis $\partial_{x_1}, \cdots, \partial_{x_N}$ are independent of $x_1, \cdots, x_m$.

If the rank of $H_1, \cdots, H_m, H_{m+1}$ is less than $m+1$, then as H_{m+1} commutes with $H_1, \cdots, H_m$, it can be written as

$$H_{m+1} = f_1\partial_{x_1} + \ldots + f_m\partial_{x_m}$$

with $\partial_{x_j} f_i = 0$ for all $1 \leq i, j \leq m$. Not all the coefficient functions f_i, $i = 1, \cdots, m$, can be constant because $H_1, \cdots, H_{m+1}$ are linearly independent, so say f_ℓ is not a constant function.

The root vector X_ℓ can be written as

$$X_\ell = \exp(x_\ell)V_\ell\,,$$

where V_ℓ is a vector field whose coefficients in the basis $\partial_{x_1}, \cdots, \partial_{x_N}$ are independent of $x_1, \cdots, x_m$.

We write

$$V_\ell = U_1 + U_2$$

where U_1 and U_2 are vector fields with

$$U_1 = \sum_{i=1}^{m} g_i\partial_{x_i} \quad \text{and} \quad U_2 = \sum_{i=m+1}^{N} g_i\partial_{x_i}$$

such that all g_i are independent of $x_1, \cdots, x_m$.

The root vector $X_\ell = \exp(x_\ell)V_\ell$ is an eigenvector for H_{m+1} with eigenvalue, say λ. Now we use the formula for Lie derivative of vector fields

$$[H,\, \exp(\chi)V] = \exp(\chi)(H(\chi)V + [H,\, V])\,.$$

Notice that any two vector fields which are combinations of $\partial_{x_1}, \cdots, \partial_{x_m}$ with coefficients that are independent of $x_1, \cdots, x_m$ actually commute. Hence

$$\begin{aligned}[H_{m+1},\, X_\ell] &= \exp(x_\ell)(H_{m+1}(x_\ell)(U_1+U_2) + [H_{m+1},\, U_1] + [H_{m+1},\, U_2]) \\ &= \exp(x_\ell)(H_{m+1}(x_\ell)(U_1+U_2) + [H_{m+1},\, U_2]) = \lambda\cdot\exp(x_\ell)(U_1+U_2)\,.\end{aligned}$$

Thus

$$H_{m+1}(x_\ell)(U_1+U_2) + [H_{m+1},\, U_2] = \lambda(U_1+U_2)\,. \tag{2.1}$$

Now H_{m+1} is a combination of $\partial_{x_1}, \cdots, \partial_{x_m}$ and U_2 is a combination of $\partial_{x_{m+1}}, \cdots, \partial_{x_N}$, and all the coefficients of both the vector fields are independent of $x_1, \cdots, x_m$. Consequently, $[H_{m+1},\, U_2]$ is a combination of $\partial_{x_1}, \cdots, \partial_{x_m}$. Taking the two sides of (2.1) modulo $\partial_{x_1}, \cdots, \partial_{x_m}$ we see that

$$H_{m+1}(x_\ell)\cdot U_2 = \lambda\cdot U_2\,.$$

But $H_{m+1}(x_\ell)$ is not a constant. Thus U_2 must be identically zero. Now (2.1) reads

$$H_{m+1}(x_\ell)\cdot U_1 = \lambda\cdot U_1\,,$$

so U_1 must also be identically zero. Consequently, X_ℓ must be identically zero. This is a contradiction. Therefore, we conclude that the rank of $H_1, \cdots, H_m, H_{m+1}$ is $m+1$.

To complete the proof, take a standard set of generators $\{X_{\alpha_i}, X_{-\alpha_i}\}_{i=1}^n$ corresponding to the simple roots $\alpha_1, \cdots, \alpha_n$. The Cartan subalgebra is spanned by

$$H_{\alpha_i} = [X_{\alpha_i}, X_{-\alpha_i}], \quad i = 1, \cdots, n.$$

As seen above, we can introduce coordinates $x_1, \cdots, x_N$ in which, because of commutativity of the $H_{\alpha_1}, \cdots, H_{\alpha_n}$ and the rank of the system of vector fields $\{H_{\alpha_1}, \cdots, H_{\alpha_n}\}$ being n, the vector field H_{α_i} becomes ∂_{x_i} for all $1 \le i \le n$.

The root vectors corresponding to the simple roots and their negatives are

$$X_{\alpha_i} = \exp(\chi_i)V_i, \quad X_{-\alpha_i} = \exp(-\chi_i)W_i,$$

where χ_i are linear functions given by $\chi_i = \sum_{j=1}^n c_{ij}x_j$ and $c_1, \cdots, c_n$ are entries of the Cartan matrix defined by the simple system of roots and the vector fields V_i, W_i, $i = 1, \cdots, n$, have coefficients that are independent of the coordinates $x_1, \cdots, x_n$.

As the Cartan matrix is nonsingular, we can make a linear change of variables $\widetilde{x}_i = \chi_i$, $i = 1, \cdots, n$, while leaving the remaining variables, if there is any, unchanged.

Thus in these variables the root vectors corresponding to the simple roots and their negatives have the form stated in the theorem.

3. Applications of Theorem 2.1

Let us apply Theorem 2.1 to determine local representations of $\mathrm{sl}(n,\mathbb{R})$ as vector fields on $\mathbb{R}^n$. This will be used in an essential way in finding invariant systems of differential equations as well as all semisimple algebras of vector fields in $\mathbb{R}^n$, with $n \le 3$.

As is well known, any semisimple Lie algebra with a split Cartan subalgebra is generated by copies of $\mathrm{sl}(2,\mathbb{R})$ – one copy for each node of the Dynkin diagram. The copies on adjacent nodes form a rank two subalgebra; moreover for root systems with only single bonds, the rank two subalgebras are copies of $\mathrm{sl}(3,\mathbb{R})$ or $\mathrm{sl}(2,\mathbb{R}) \times \mathrm{sl}(2,\mathbb{R})$.

Thus, it is important to determine representations of subalgebras as vector fields on $\mathbb{R}^N$ of the type specified in the following proposition.

Proposition 3.1. *Any representation of* $\mathrm{sl}(3,\mathbb{R})$ *as vector fields on* $\mathbb{R}^N$ *whose root spaces for the simple roots are of the form* $X_\alpha = \exp(x)U$, $X_\beta = \exp(y)V$, *where* U *and* V *are constant vector fields must be – up to multiplicative non-zero constants – of the form*

$$X_\alpha = \exp(x)(\partial_x + U_1), \quad X_\beta = \exp(y)(\partial_x - \partial_y + V_1)$$

or

$$X_\alpha = \exp(x)(\partial_x - \partial_y + U_1), \quad X_\beta = \exp(y)(\partial_y + V_1),$$

where U_1, V_1 *are constant vector fields whose* ∂_x *and* ∂_y *components vanish for the basis*

$$\partial_x = \partial_{x_1}, \partial_y = \partial_{x_2}, \partial_{x_3}, \cdots, \partial_{x_N}.$$

Proof. For vector fields U, V and functions χ, ψ,

$$[\exp(\chi)U, \exp(\psi)V] = \exp(\chi+\psi)(U(\psi)\cdot V - V(\chi)\cdot U + [U, V]).$$

In particular, if U and V are constant vector fields on $\mathbb{R}^N$, then

$$[\exp(\chi)U,\, \exp(\psi)V] \;=\; \exp(\chi+\psi)(U(\psi)\cdot V - V(\chi)\cdot U)\,.$$

First note that if $X = \exp(x)A$, $Y = \exp(-x)B$ generate $\mathrm{sl}(2,\mathbb{R})$, with X, Y eigenvectors for their commutator with opposite eigenvalues, and

$$A \;=\; a\partial_x + C\,, \quad B \;=\; c\partial_x + D$$

are constant vector fields, with C, D supported outside ∂_x, then both a and c must be nonzero, otherwise X, Y would generate a solvable algebra. By "supported outside ∂_x" we mean that the expression does not contain the term ∂_x; we will employ this terminology.

Using the notation of the statement of the proposition, write

$$U \;=\; a\partial_x + b\partial_y + U_1\,, \quad V \;=\; c\partial_x + d\partial_y + V_1\,,$$

where U_1, V_1 are constant vector fields supported outside ∂_x, ∂_y; thus they play no role in the commutation relations of the Lie algebra generated by X_α and X_β. Therefore, to prove the proposition, we may ignore them. As remarked above, we may assume that a, c are both non-zero and therefore we may assume

$$X_\alpha \;=\; \exp(x)(\partial_x + \lambda\cdot\partial_y + U_1)\,, \quad X_\beta \;=\; \exp(y)(\mu\cdot\partial_x + \partial_y + V_1)\,.$$

The commutation relations are not affected by ignoring the constant vector fields supported outside ∂_x, ∂_y and we ignore them henceforth. So we may assume that

$$X_\alpha \;=\; \exp(x)(\partial_x + \lambda\cdot\partial_y)\,, \quad X_\beta \;=\; \exp(y)(\mu\cdot\partial_x + \partial_y)\,.$$

The positive roots of a system of type A_2 are α, β, $\alpha+\beta$. Hence $[X_\alpha,\, X_\beta]$ commutes with both X_α, X_β. Now

$$[X_\alpha,\, X_\beta] \;=\; \exp(x+y)((\lambda-1)\mu\partial_x + (1-\mu)\lambda\partial_y)\,. \tag{3.1}$$

Its commutator with X_α is:

$$\exp(2x+y)(\lambda-1)\mu(\partial_x+\lambda\partial_y) - (\lambda+1)((\lambda-1)\mu\partial_x + (1-\mu)\lambda\partial_y))\,.$$

Hence $[X_{\alpha+\beta},\, X_\alpha] = 0$ implies that

$$(\lambda-1)\mu - (\lambda-1)(\lambda+1)\mu \;=\; 0 \;=\; (\lambda-1)\lambda\mu - (\lambda+1)\lambda(1-\mu)\,.$$

If both λ and μ are nonzero, then these equations give $\lambda = 1 = \mu$. But then

$$[X_\alpha,\, X_\beta] \;=\; 0$$

from (3.1).

Thus one of λ and μ is zero. If $\lambda = 0$, we may take $X_\alpha = \exp(x)\partial_x$. Consequently,

$$X_{\alpha+\beta} \;=\; \exp(x+y)(\lambda-1)\mu\partial_x\,,$$

and we may therefore take $X_{\alpha+\beta} = \exp(x+y)\partial_x$.

Using

$$[X_\alpha,\, X_{\alpha+\beta}] \;=\; \exp(x+2y)(\mu+1)\partial_x \;=\; 0$$

gives $\mu = -1$. This yields the first representation

$$X_\alpha \;=\; \exp(x)\partial_x\,, \quad X_\beta \;=\; \exp(y)(\partial_x - \partial_y)\,.$$

If $\mu = 0$, then we may take $X_\beta = \exp(y)\partial_y$. Therefore, from (3.1),

$$[X_\alpha, X_\beta] = \exp(x+y)\lambda\partial_y .$$

Therefore, $\lambda \neq 0$, and we can take

$$X_{\alpha+\beta} = \exp(x+y)\partial_y .$$

Its commutator with X_β is zero, while its commutator with X_α is $\exp(2x+y)(1+\lambda)\partial_y$. As this commutator is zero, we must have $\lambda = -1$. This gives the second representation

$$X_\alpha = \exp(x)(\partial_x - \partial_y) , \quad X_\beta = \exp(y)\partial_y .$$

This completes the proof. □

Corollary 3.2. *Any representation of* $\mathrm{sl}(n+1,\mathbb{R})$ *as vector fields on* $\mathbb{R}^n$, $n \geq 2$, *is equivalent by point transformations to the two representations given by the following root vectors for the simple roots and their negatives and both the representations are equivalent by a point transformation:*

(1) $X_1 = \exp(x_1)\partial_{x_1}$, $X_{\alpha_i} = \exp(x_i)(\partial_{x_i} - \partial_{x_{i-1}})$, $2 \leq i \leq n$,
$X_{-\alpha_i} = \exp(-x_i)(\partial_{x_i} - \partial_{x_{i+1}})$, $1 \leq i \leq n-1$, $X_{-\alpha_n} = \exp(-x_n)\partial_{x_n}$.

(2) $X_{\alpha_i} = \exp(x_i)(\partial_{x_i} - \partial_{x_{i+1}})$, $1 \leq i \leq n-1$, $X_{\alpha_n} = \exp(x_n)\partial_{x_n}$,
$X_{-\alpha_1} = \exp(-x_1)\partial_{x_1}$, $X_{-\alpha_i} = \exp(-x_i)(\partial_{x_i} - \partial_{x_{i-1}})$, $2 \leq i \leq n$.

Proof. Use Proposition 3.1 and Theorem 2.1 inductively for the chain of subalgebras

$$\mathrm{sl}(2,\mathbb{R}) \subset \mathrm{sl}(3,\mathbb{R}) \subset \cdots \subset \mathrm{sl}(n,\mathbb{R}) \subset \mathrm{sl}(n+1,\mathbb{R})$$

and the commutator relations $[X_{-r}, X_s] = 0$ for simple and unequal roots r, s and $[X_r, X_s] = 0$ if $r+s$ is not a root and $r \neq -s$. □

Corollary 3.3.

(1) *Real analytic semisimple algebras of vector fields in* $\mathbb{R}$ *can be only split and of type* A_1.
(2) *Real analytic semisimple Lie algebras of vector fields in* $\mathbb{R}^2$ *can only be real forms of algebras of types* A_1, $A_1 \times A_1$ *or* A_2.
(3) *Real analytic semisimple Lie algebras of vector fields in* $\mathbb{R}^3$ *– apart from the types listed in (2) – can only be real forms of algebras of types* B_2, $A_2 \times A_1$ *or* A_3.

The proof of this corollary together with all the real forms will appear elsewhere.

4. Applications to systems of ordinary differential equations

As already mentioned in the introduction, to find linearizable equations, one could look at systems of ordinary differential equations with known semisimple Lie algebras of symmetries and find their realizations as vector fields and determine the joint invariants in a suitable prolongation.

Here, we restrict ourselves to the algebras $\mathrm{sl}(3,\mathbb{R})$ and $\mathrm{sl}(4,\mathbb{R})$ that are known to be the symmetry algebras of $y'' = 0$ and $y'' = 0 = z''$, respectively. Therefore, we need to find all representations of $\mathrm{sl}(3,\mathbb{R})$ and $\mathrm{sl}(4,\mathbb{R})$ as vector fields on $\mathbb{R}^3$ and $\mathbb{R}^4$. Declaring one of the variables to be the independent variable and considering the remaining as dependent, we need to find the invariants in the second prolongation. If, in the canonical coordinates for an algorithmically computable abelian subalgebra there are linear systems present, then those coordinates would give the linearizing coordinates.

We will implement this program in this section.

Proposition 4.1.

(1) *If a second order ordinary differential equation of the form* $y'' = f(x, y, y')$ *has* $\mathrm{sl}(3, \mathbb{R})$ *as its symmetry algebra, then the canonical coordinates for the maximal abelian subalgebra of ad-nilpotent elements of maximal rank give the linearizing coordinates for the given ordinary differential equation and in these coordinates the equation is* $y'' = 0$*.*

(2) *If a system of second order ordinary differential equations of the form* $y''(x) = f(x, y, z, y', z')$*,* $z''(x) = g(x, y, z, y', z')$ *has* $\mathrm{sl}(4, \mathbb{R})$ *as its symmetry algebra, then the canonical coordinates for the maximal abelian subalgebra of ad-nilpotent elements of maximal rank give the linearizing coordinates for the given system of ordinary differential equations. In these coordinates the system becomes* $y''(x) = 0 = z''(x)$*.*

Before giving a proof of Proposition 4.1, we recall, for the benefit of non-specialists, how on can compute prolongations of vector fields ab initio – following Lie [Li1, p. 261–274]. For generalities, the reader is referred to [Ib] and [Ol].

First of all, a vector field in local coordinates $(x_1, \cdots, x_n)$ is a sum of vector fields $f_i\partial_{x_i}$. Therefore one needs to only find prolongations of such fields and add them to get the prolongation to any desired order. Secondly, the prolongations are obtained by repeated use of the chain rule.

Bearing this in mind, the proof of Proposition 4.1 is a consequence of the canonical representations of $\mathrm{sl}(3, \mathbb{R})$ and $\mathrm{sl}(4, \mathbb{R})$. Although the first part is already in [Li1], we have given details in this case also as the same ideas regarding linearizing coordinates and where they live work also for systems of ordinary differential equations.

5. Proof of Proposition 4.1

5.1. **Proof of part (1).** Let $L = \mathrm{sl}(3, \mathbb{R})$. It has only one representation – up to point transformations – given by

$$X_\alpha = \exp(x)(\partial_x - \partial_y),\ X_\beta = \exp(y)\partial_y,\ X_{-\alpha} = \exp(-x)\partial_x,\ X_{-\beta} = \exp(-y)(\partial_x - \partial_y)\,.$$

A maximal ad-nilpotent subalgebra corresponds to all the root spaces for the positive roots. Thus, besides X_α and X_β, it has, as a basis, $X_{\alpha+\beta} = \exp(x + y)\partial_y$. The algebra generated by X_α and $X_{\alpha+\beta}$ is abelian and its rank is two. It is a maximal abelian algebra made up of root vectors. We write the full algebra in terms of the canonical coordinates for this abelian algebra.

The canonical coordinates for this abelian algebra are given by $\exp(x)(\partial_x - \partial_y) = \partial_u$, $\exp(x + y)\partial_y = \partial_v$. The differentials of u, v are dual to the fields ∂_u, ∂_v. Thus, in these coordinates, the root spaces work out to be

$$X_\alpha = \partial_u,\ X_\beta = u\partial_v,\ X_{-\alpha} = u(u\partial_u + v\partial_v),\ X_{-\beta} = v\partial_u\,.$$

For convenience, we write u, v as x, y respectively.

As vector fields on a plane,

$$X_\alpha = (1,\, 0),\ X_\beta = (0,\, x),\ X_{-\alpha} = (x^2,\, xy),\ X_{-\beta} = (y,\, 0)\,.$$

The corresponding flows – up to order ϵ – are given by:

$$(\widetilde{x},\, \widetilde{y}) = (x+\epsilon,\, y),\ (\widetilde{x},\, \widetilde{y}) = (x,\, y+\epsilon x),\ (\widetilde{x},\, \widetilde{y}) = (x+\epsilon x^2,\, y+\epsilon xy),\ (\widetilde{x},\, \widetilde{y}) = (x+\epsilon y,\, y)\,.$$

To find the prolonged action, we need to declare one of the variables as independent and the remaining variables as dependent and then we have to look at how the first and second derivatives are transformed by these flows. If

$$(\widetilde{x},\, \widetilde{y}) = (x+\epsilon,\, y)\,,$$

then

$$\frac{d\widetilde{y}}{d\widetilde{x}} = \frac{dy}{dx} \quad \text{and} \quad \frac{d^2\widetilde{y}}{d\widetilde{x}^2} = \frac{d^2y}{dx^2}\,.$$

Consequently, the second prolongation of X_α has no components in the variables y', y'' and therefore it is given by $X_\alpha^{(2)} = \partial_x$.

If $(\widetilde{x},\, \widetilde{y}) = (x,\, y+\epsilon x)$, then

$$\frac{d\widetilde{y}}{d\widetilde{x}} = \frac{dy+\epsilon dx}{dx} = \frac{dy}{dx} + \epsilon\,,\quad \frac{d^2\widetilde{y}}{d\widetilde{x}^2} = \frac{d}{d\widetilde{x}}(\frac{d\widetilde{y}}{d\widetilde{x}}) = \frac{\frac{d}{dx}(\frac{dy}{dx}+\epsilon)}{\frac{d\widetilde{x}}{dx}} = \frac{d^2y}{dx^2}\,,$$

therefore $X_\beta^{(2)} = x\partial_y + \partial_{y'}$.

For computing the second prolongation of $X_{-\alpha} = x(x\partial_x + y\partial_y)$, it is convenient to compute the second prolongations of $x^2\partial_x$ and $xy\partial_y$ and add them. Thus if $(\widetilde{x},\, \widetilde{y}) = (x+\epsilon x^2,\, y)$, then

$$\frac{d\widetilde{y}}{d\widetilde{x}} = \frac{dy}{dx} - 2\epsilon x\frac{dy}{dx} + \cdots\,,\quad \frac{d^2\widetilde{y}}{d\widetilde{x}^2} = \frac{d^2y}{dx^2} - 2\epsilon(\frac{dy}{dx} + x\frac{d^2y}{dx^2}) - 2\epsilon x\frac{d^2y}{dx^2} + \cdots\,.$$

Therefore, the second prolongation of $x^2\partial_x$ is

$$x^2\partial_x - 2xy'\partial_{y'} - 2(y' + 2xy'')\partial_{y''}\,. \tag{5.1}$$

If $(\widetilde{x},\, \widetilde{y}) = (x,\, y+\epsilon xy)$, then

$$\frac{d\widetilde{y}}{d\widetilde{x}} = \frac{dy}{dx} + \epsilon(y + x\frac{dy}{dx})\,,\quad \frac{d^2\widetilde{y}}{d\widetilde{x}^2} = \frac{d^2y}{dx^2} + \epsilon(2\frac{dy}{dx} + x\frac{d^2y}{dx^2})\,.$$

Hence the second prolongation of $xy\partial_y$ is

$$xy\partial_y + (y + xy')\partial_{y'} + (2y' + xy'')\partial_{y''}\,. \tag{5.2}$$

Adding (5.1) and (5.2) gives the second prolongation of $X_{-\alpha}$ namely,

$$X_{-\alpha}^{(2)} = x^2\partial_x + xy\partial_y + (y - xy')\partial_{y'} - 3xy''\partial_{y''}\,.$$

Now suppose we have a second order equation of the form $y'' = f(x, y, y')$ – invariant under X_α and X_β; this means that it is at least invariant under $X_\alpha^{(2)} = \partial_x$ and $X_\beta^{(2)} = x\partial_y + \partial_{y'}$. Then it is also invariant under $[X_\alpha^{(2)},\, X_\beta^{(2)}] = \partial_y$. Thus the equation is of the form $y'' = f(y')$. Hence invariance under $X_\beta^{(2)}$ shows that the equation is $y'' - k = 0$, where k is a constant. Applying $X_{-\alpha}^{(2)}$ to the equation we must have $-3xy'' = 0$ on the hypersurface $y'' - k = 0$. Therefore $k = 0$ and the equation is invariant under the full algebra.

Hence the only second order equation with symmetry algebra $\mathrm{sl}(3,\mathbb{R})$ is – in suitable coordinates $y'' = 0$. These coordinates are the canonical coordinates of the maximal abelian subalgebra of maximal rank in the nil-radical of a Borel subalgebra of the symmetry algebra $\mathrm{sl}(3,\mathbb{R})$. The nil-radical of the Borel subalgebra can be algorithmically determined from a single ad-nilpotent element as shown in [AABGM]. This completes the proof of the first part.

5.2. **Alternative proof of part (1).** We note that while an equation $y'' = f(x,y,y')$ is transformed to a similar equation under point transformations, this is no longer true for a system of two such equations. We give below an alternative argument for part (1) whose ideas work also for systems.

A maximal ad-nilpotent subalgebra corresponds to all the root spaces for the positive roots. Thus, besides X_α and X_β, it has, as a basis, $X_{\alpha+\beta} = \exp(x+y)\partial_y$. The algebra generated by X_α and $X_{\alpha+\beta}$ is abelian and of rank two. It is a maximal abelian algebra A made up of root vectors. We write the full algebra in terms of the canonical coordinates $(x,\, y)$ for this abelian algebra.

In the representation of $\mathrm{sl}(3,\mathbb{R})$ in $V(\mathbb{R}^2)$ extended to the 2nd prolongation, we have independent variables x, y, y', y''. Then, by the Implicit function theorem (see, e.g. [dO]) the invariant nonsingular hypersurfaces H have defining equations in which – because of invariance under ∂_x, ∂_y – the variables x, y do not appear. Thus the equations for H can only be

$$y'' - f(y') = 0 \quad \text{or} \quad y' - g(y'') = 0\,.$$

The second equation cannot occur because of the vector field $X_\beta = x\partial_y$ whose second prolongation is $x\partial_y + \partial_{y'}$. Consequently, we have only one possibility $y'' - f(y') = 0$.

Applying $x\partial_y + \partial_{y'}$ we have $f(y') = k$. Hence the equation of H is $y'' - k = 0$.

Applying $X^{(2)}_{-\alpha} = x^2\partial_x + xy\partial_y + (y - xy')\partial_{y'} - 3xy''\partial_{y''}$ to the equation gives $xy'' = 0$ and $y'' = k$. Hence if $k \neq 0$, then the function x would vanish identically on H and this contradicts that H is a hypersurface. Thus the only possibility is that H is defined by $y'' = 0$.

Finally, as $X^{(2)}_{-\beta} = y\partial_x - (y')^2\partial_{y'} - 3y'y''\partial_{y''}$, it follows that H is also invariant under $X^{(2)}_{-\beta}$. Therefore as H is invariant under the generators $X^{(2)}_{\alpha}$, $X^{(2)}_{-\alpha}$, $X^{(2)}_{\beta}$ and $X^{(2)}_{-\beta}$, it is invariant under $\mathrm{sl}(3,\mathbb{R})$.

5.3. **Proof of part (2).** As the form of invariant systems is not in general invariant under point transformations, we need to approach this problem in a more geometric way, combined with detailed information on the root vectors and their prolongations.

First of all we note that in the prolonged space with coordinates x, y, z, y', z', y'', z'' if we have a system of equations

$$F(x,y,z,y',z',y'',z'') = 0 = G(x,y,z,y',z',y'',z'')$$

and the rank is two at a point p of the subset $F = 0 = G$, then locally, the set M defined by these equations is a five dimensional submanifold of the extended space with coordinates x, y, z, y', z', y'', z''.

By the implicit function theorem, the condition of rank being two near a given point p of M implies that we can solve for two of the variables explicitly in terms of the remaining variables.

If M is invariant under translations ∂_x, ∂_y, ∂_z, then none of these variables can be x, y or z. Thus we have to choose two of the variables from y', y'', z', z''. Moreover, the resulting equations cannot involve x, y or z as free variables, again because of invariance under ∂_x, ∂_y, ∂_z.

If M is also invariant under $X = x\partial_y$ then as the second prolongation of X is $x\partial_y + \partial_{y'}$, none of these equations can have y' as an independent variable or a dependent variable.

So we have the following possibilities:

(i) Dependent variables are y'', z', independent variable is z'' – and equations are $y'' = f(z'')$, $z' = g(z'')$.
(ii) Dependent variables are z', z'', independent variable is y'' – and equations are $z' = f(y'')$, $z'' = g(y'')$.
(iii) Dependent variables are y'', z'', independent variable is z' – and equations are $y'' = f(z')$, $z'' = g(z')$.

To determine these equations in the case at hand, we need detailed information about the root vectors and their second prolongations.

We choose coordinates adapted to the structure of the given Lie algebra. In the case of $\mathrm{sl}(4,\mathbb{R})$ the coordinates representing one independent and two dependent variables will be the canonical coordinates of a canonically defined subalgebra, uniquely determined up to conjugation.

By Corollary 3.2, the Lie algebra $L = \mathrm{sl}(4,\mathbb{R})$ has only one representation as vector fields in three dimensions; up to point transformations it is given by given by the root vectors

$$X_\alpha = \exp(x)\partial_x,\ X_\beta = \exp(y)(\partial_y - \partial_x),\ X_\gamma = \exp(z)(\partial_z - \partial_y)$$
$$X_{-\alpha} = \exp(-x)(\partial_x - \partial_y),\ X_{-\beta} = \exp(-y)(\partial_y - \partial_z),\ X_{-\gamma} = \exp(-z)\partial_z\,.$$

A maximal abelian subalgebra of ad-nilpotent elements of geometric rank 3 is

$$X_{\alpha+\beta+\gamma} = \exp(x+y+z)\partial_x,\ X_{\beta+\gamma} = \exp(y+z)(\partial_y - \partial_x),\ X_\gamma = \exp(z)(\partial_z - \partial_y)\,.$$

The canonical coordinates for this algebra are given by solving

$$\exp(x+y+z)\partial_x = \partial_u,\ \exp(y+z)(\partial_y - \partial_x) = \partial_v,\ \exp(z)(\partial_z - \partial_y) = \partial_w\,.$$

Solving this system, we have

$$u = -\exp(-x-y-z),\ v = -\exp(-y-z),\ w = -\exp(-z)\,.$$

Thus $\partial_x = -u\partial_u$, $\partial_y - \partial_x = -v\partial_v$ and $\partial_z - \partial_y = -w\partial_w$. Also

$$\exp(z) = -\frac{1}{w},\ \exp(y) = \frac{w}{v},\ \exp(x) = \frac{v}{u}\,.$$

Therefore, in these coordinates, ignoring signs,

$$X_\alpha = v\partial_u,\ X_\beta = w\partial_v,\ X_\gamma = \partial_w,\ X_{-\alpha} = u\partial_v$$
$$X_{-\beta} = v\partial_w,\ X_{-\gamma} = w(u\partial_u + v\partial_v + w\partial_w)\,.$$

We need to compute the second prolongation of these fields. To do this, we have to declare one of these variables as the independent and the remaining variables as the dependent variables.

We take u as the independent variable and v, w as the dependent variables.

Following conventions, we re-label $u = x$, $v = y$ and $w = z$. Thus, our fields are

$$X_\alpha = y\partial_x,\ X_\beta = z\partial_y,\ X_\gamma = \partial_z,\ X_{-\alpha} = x\partial_y$$

$$X_{-\beta} = y\partial_z,\ X_{-\gamma} = z(x\partial_x + y\partial_y + z\partial_z)\,.$$

We find the second prolongations by using the chain rule as in [Li1, p. 261–274]. We obtain

- $X_\alpha^{(2)} = y\partial_x - (y')^2\partial_{y'} - y'z'\partial_{z'} - 3y'y''\partial_{y''} - (y''z' + 2y'z'')\partial_{z''}$,
- $X_\beta^{(2)} = z\partial_y + z'\partial_{y'} + z''\partial_{y''}$,
- $X_{-\alpha}^{(2)} = x\partial_y + \partial_{y'}$,
- $X_{-\beta}^{(2)} = y\partial_z + y'\partial_{z'} + y''\partial_{z''}$,
- $X_\gamma^{(2)} = \partial_z$.

To find $X_{-\gamma}^{(2)}$, it is convenient to find the second prolongations of $zx\partial_x$, $zy\partial_y$, $z^2\partial_z$ and add them. We have:

- $(zx\partial_x)^{(2)} = zx\partial_x - y'(xz'+z)\partial_{y'} - z'(xz'+z)\partial_{z'} - (2y''(xz'+z)+y'(xz''+2z'))\partial_{y''} - (3xz'z'' + 2(z')^2 + 2zz'')\partial_{z''}$,
- $(zy\partial_y)^{(2)} = yz\partial_y + (y'z + yz')\partial_{y'} + (y''z + 2y'z' + yz'')\partial_{y''}$,
- $(z^2\partial_z)^{(2)} = z^2\partial_z + 2zz'\partial_{z'} + 2((z')^2 + zz'')\partial_{z''}$.

We can now determine all the invariant systems. We have the following possibilities:

Case (1): Dependent variables are y'', z', independent variable is z'', and equations are $y'' = f(z'')$, $z' = g(z'')$.

Applying $X_{-\beta}^{(2)} = y\partial_z + y'\partial_{z'} + y''\partial_{z''}$, we must have $y' = y''g'(z'')$. Thus on M we have one more functionally independent equation $y' = f(z'')g'(z'')$ and dimension of M would decrease. Hence this case does not arise.

Case (2): The dependent variables are z', z'', the independent variable is y'', and the equations are $z' = f(y'')$, $z'' = g(y'')$. Applying $X_{-\beta}^{(2)} = y\partial_z + y'\partial_{z'} + y''\partial_{z''}$ to $z' = f(y'')$ gives $y' = 0$ along the solution space M and its dimension would decrease. Thus, this case also does not arise.

Case (3): The dependent variables are y'', z'', the independent variable is z', and the equations are $y'' = f(z')$, $z'' = g(z')$. Applying $X_{-\beta}^{(2)} = y\partial_z + y'\partial_{z'} + y''\partial_{z''}$ we have $y'f'(z') = 0$, $y'' = y'g'(z')$.

If $f'(z')$ is not identically zero along the solution space then $y' = 0$ and the dimension of M would decrease. Hence $f'(z') = 0$ along M.

Thus $y'' = k$ along M and $k = y'g'(z')$. If $k = 0$, then y' cannot vanish along M and therefore $y'g'(z') = 0$ implies $g(z') = l$ and M is defined by $y'' = 0$, $z'' = l$. If $k \neq 0$, then we would have an extra equation

$$y' = \frac{k}{g'(z')}$$

and the dimension of M would go down.

Consequently, the only possibility for M is that it is defined by $y'' = 0$, $z'' = l$.

Applying $X_\alpha^{(2)}$ to $z'' = l$ we get $-2y'z'' = 0$ on M and if y' is identically zero on some open set of M, then again we would get an independent equation $y' = 0$ and the local equations for M would be $y'' = 0$, $z'' = l$, $y' = 0$ and the dimension of M would go down. Thus, z'' is identically 0 on M and the equations for M are indeed $y'' = 0 = z''$.

It remains to check that this is indeed an invariant submanifold of $\mathrm{sl}(4, \mathbb{R})$ in these coordinates. From the equations for the second prolongations of the generators, it only remains to check invariance under $X_{-\gamma}^{(2)}$.

We have

$$X_\alpha^{(2)} = y\partial_x - (y')^2\partial_{y'} - y'z'\partial_{z'} - 3y'y''\partial_{y''} - (y''z' + 2y'z'')\partial_{z''}\,, \quad X_\gamma^{(2)} = \partial_z$$

$$X_{-\alpha}^{(2)} = x\partial_y + \partial_{y'}\,, \quad X_{-\beta}^{(2)} = y\partial_z + y'\partial_{z'} + y''\partial_{z''}$$

and $X_{-\gamma}^{(2)}$ is a sum of fields and we need to just consider the contributions in the $\partial_{y''}$ and $\partial_{z''}$ directions; these contributions are:

$$-(2y''(xz' + z) + y'(xz'' + 2z'))\partial_{y''} - (3xz'z'' + 2(z')^2 + 2zz'')\partial_{z''}\,,$$

$$(y''z + 2y'z' + yz'')\partial_{y''} \quad \text{and} \quad 2((z')^2 + zz'')\partial_{z''}$$

from the formulas given above; their sum vanishes on the set $y'' = 0 = z''$. This completes the proof of the proposition.

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