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ABSTRACT. A constructive method, based on Cartan's theory of highest weights, for decomposing finite dimensional representations of real Lie algebras is developed. The procedure is illustrated by numerous examples.

1. INTRODUCTION

The theory of finite dimensional real representations of real semisimple Lie algebras — due to Cartan, Iwahori and Karpelevich — is explained in detail in books [GG], [On]. The relevant computational aspects of complex representations of complex semisimple Lie algebras are dealt with in [deG1]. This is essentially a problem in linear algebra, as shown also in [CS, p. 189, Theorem 2.2.14].

In this paper, we give a constructive procedure for decomposing the complexification of a given representation of a real semisimple Lie algebra \mathfrak{g} into \mathfrak{g} -invariant and conjugation invariant summands, which are either irreducible or sums of two irreducible summands. This is done using Cartan's theory of highest weights. There are essentially three cases:

- (a) the irreducible component M is self conjugate,
- (b) M and its conjugate have highest weight vectors with different weights, and
- (c) M and its conjugate have highest weight vectors with the same weight and the highest weight vectors are linearly independent.

We also give the \mathfrak{g} -irreducible components of real points in these components.

As the algorithms for complex representations are essential for the real case, we have also discussed them briefly.

The main ideas needed for this paper from algorithmic Lie theory are reviewed in Section 2.1, which itself is based on the paper [AABGM]. As general references for Lie theory, the reader is referred to [HN], [Kn] and to [GG], [On] for representation theory of real Lie algebras. For algebraic groups, the reader is referred to [St1], [deG2].

We should point out that the main technical point of the paper is the algorithmic determination of the image — under conjugation — of any Borel subalgebra of $L^{\mathbb{C}}$, where L is a semisimple subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, that contains the complexification of a given Cartan subalgebra of L (cf. Lemma 4.2 and Corollary 4.3) — and an algorithmic identification

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of weight and weight vectors and their conjugates. These results allow for a constructive decomposition of real representations, of course within the computational capabilities of software, into real irreducible components.

Some recent applications of the theory of highest weights to physics are [AKHvD], [SHK].

2. PRELIMINARIES AND NOTATION

Throughout this paper, $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ will denote a semisimple Lie algebra and G the subgroup of $GL(n, \mathbb{R})$ with Lie algebra \mathfrak{g} . The complexification of G , denote by $G^{\mathbb{C}}$ is the subgroup of $GL(n, \mathbb{C})$ with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$.

The group G is generated by real one-parameter subgroups

$$\{\exp(rX) \mid r \in \mathbb{R}\}, \quad X \in \mathfrak{g},$$

while $G^{\mathbb{C}}$ is generated by the complex 1-parameter subgroups

$$\{\exp(zX) \mid z \in \mathbb{C}\}, \quad X \in \mathfrak{g}.$$

The connected component of the real points of $G^{\mathbb{C}}$ containing the identity element is the group G . If ρ is a representation of \mathfrak{g} in $\mathfrak{gl}(V)$, where V is a finite dimensional real vector space, its complexification $\rho^{\mathbb{C}}$ is the representation of $\mathfrak{g}^{\mathbb{C}}$ in $\mathfrak{gl}(V^{\mathbb{C}})$, where $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \sqrt{-1}V$: it is defined by $\rho^{\mathbb{C}}(X + \sqrt{-1} \cdot Y) = \rho(X) + \sqrt{-1} \cdot \rho(Y)$, with $X, Y \in \mathfrak{g}$.

The algebra $\mathfrak{g}^{\mathbb{C}}$ has a conjugation

$$\sigma : \mathfrak{g}^{\mathbb{C}} \longrightarrow \mathfrak{g}^{\mathbb{C}}, \quad X + \sqrt{-1} \cdot Y \longmapsto X - \sqrt{-1} \cdot Y.$$

Similarly $V^{\mathbb{C}}$, $\mathfrak{gl}(V^{\mathbb{C}})$ and $G^{\mathbb{C}}$ have conjugations defined in the obvious way. We will denote all of them by the same symbol σ , and occasionally write $\sigma(v) = \bar{v}$ etc.

2.1. Roots. We will follow the definition of roots given in [AABGM].

Let \mathfrak{c} be a Cartan subalgebra of \mathfrak{g} . By definition, \mathfrak{c} is a maximal abelian subalgebra of \mathfrak{g} which is diagonalizable in the complexification $\mathfrak{g}^{\mathbb{C}}$ of \mathfrak{g} . A nonzero vector $v \in \mathfrak{g}^{\mathbb{C}}$ such that $[h, v] = \lambda(h)v$ for all h in \mathfrak{c} is called a root vector and the corresponding linear function λ a root of the Cartan subalgebra \mathfrak{c} .

A complex number $z = a + \sqrt{-1} \cdot b$ with $a, b \in \mathbb{R}$ is said to be positive if either $a > 0$ or $a = 0$ and $b > 0$.

Fixing a basis in \mathfrak{c} defines a cone in the set of complex valued linear functions on \mathfrak{c} : namely, if we fix an ordered basis h_1, \dots, h_r of \mathfrak{c} , a nonzero complex valued linear function λ on \mathfrak{c} is said to be positive if the first nonzero complex number among $\lambda(h_1), \dots, \lambda(h_r)$ is a positive complex number. Otherwise it is called negative. Positive roots which are not a sum of two positive roots are called simple roots. Thus a knowledge of all positive roots determines algorithmically the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$.

In general, any Borel subalgebra determines a system of positive roots — without any reference to a basis of a given Cartan subalgebra of the Borel subalgebra. This happens typically when one embeds a given subalgebra of ad-nilpotent elements into a maximal subalgebra of such elements, following the algorithms in [AABGM]. An example is given in section 5 of [AABGM], where a system of type G_2 in \mathbb{R}^2 with rational coordinates comes up when one embeds a commuting algebra of translations into a maximal ad-nilpotent subalgebra.

3. COMPLEX REPRESENTATIONS OF COMPLEX SEMISIMPLE LIE ALGEBRAS

In this section we review briefly the theory of highest weight vectors [HN, p. 181–187], [Kn, p. 225–229].

Let \mathfrak{g} be a complex semisimple subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ and \mathfrak{c} a Cartan subalgebra of \mathfrak{g} . A positive system of roots, for example one determined by fixing a basis of \mathfrak{c} , determines a Borel subalgebra $\mathfrak{b} = \mathfrak{c} + \mathfrak{n}$, where \mathfrak{n} is spanned by all root vectors corresponding to the chosen system of positive roots.

Let ρ be a complex representation of \mathfrak{g} in $\mathfrak{gl}(V)$, where V is a finite dimensional complex vector space and G_ρ the subgroup of $\text{GL}(V)$ whose Lie algebra is $\rho(\mathfrak{g})$. This group G_ρ is an algebraic subgroup of $\text{GL}(V)$.

Definition 3.1. A nonzero vector $v \in V$ is called a *high weight* vector if $n.v = 0$ for all $n \in \mathfrak{n}$ and $h(v) = \lambda(h)v$ for all h in the Cartan subalgebra \mathfrak{c} . The weight λ is then called a *high weight*.

We will need a version of the highest weight theorem in the following form:

Theorem 3.2.

- (a) *If ρ is a representation of \mathfrak{g} in $\mathfrak{gl}(V)$, and w is a high weight vector, then the linear span $\langle G_\rho w \rangle$ (of $G_\rho w$) is irreducible. Moreover, if B is the Borel subgroup of G_ρ with Lie algebra \mathfrak{b} , and U^- is the unipotent radical of the opposite Borel subgroup, then $\langle G_\rho w \rangle$ is the linear span $\langle U^- w \rangle$ of $U^- w$.*
- (b) *If ρ is an irreducible representation of \mathfrak{g} in $\mathfrak{gl}(V)$, then there is a unique line in V fixed by the Borel subalgebra \mathfrak{b} .*

Proof. See [St2, p. 83]. The main point of the proof is that the big cell $U^- B$ is Zariski dense in G_ρ . \square

The proof of the following corollary depends on the complete reducibility of representations of complex semisimple Lie algebras.

Corollary 3.3.

- (a) *If v_1, \dots, v_k are linearly independent high weight vectors in V , then the sum*

$$\langle G_\rho v_1 \rangle + \dots + \langle G_\rho v_k \rangle$$

is in fact a direct sum.

- (b) If v_1, \dots, v_k is a maximal set of linearly independent high weight vectors in V , then

$$V = \langle G_\rho v_1 \rangle + \dots + \langle G_\rho v_k \rangle$$

is the decomposition of V into its \mathfrak{g} -invariant irreducible components.

Theorem 3.2 and Corollary 3.3 together give the algorithm described below.

3.1. Algorithm for computing irreducible components. Compute the null space of η in V . As the Cartan algebra \mathfrak{c} normalizes η , the null space of η in V decomposes into common eigen-spaces for \mathfrak{c} . All the weights arising in this manner are high weights, and moreover a basis of weight vectors in the null space of η in V gives the decomposition of V into its irreducible components.

The irreducible component with highest weight vector v is $\langle G_\rho v \rangle$, the span of $G_\rho v$.

Example : Consider the standard action of $\mathfrak{sl}(n, \mathbb{C})$ on \mathbb{C}^n . Each element $A = (a_{ij})$ of $\mathfrak{sl}(n, \mathbb{C})$ defines a vector field V_A on \mathbb{C}^n , namely $V_A(p) = Ap$, where p is a column vector. We identify V_A with the operation of taking the directional derivative in the direction of V_A . This gives the identification of V_A with the differential operator $\sum_{i,j=1}^n a_{ij} x_j \frac{\partial}{\partial x_i}$. The differential operators

$$x_i \frac{\partial}{\partial x_{i+1}}, \quad i = 1, \dots, n-1,$$

generate a maximal ad nilpotent subalgebra of $\mathfrak{sl}(n, \mathbb{C})$: denote it by \mathfrak{n} . This subalgebra is normalized by the maximal torus \mathfrak{c} which is generated by

$$x_i \frac{\partial}{\partial x_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}}, \quad i = 1, \dots, n-1.$$

The analytic functions of the n variables x_1, \dots, x_n annihilated by \mathfrak{n} are functions of x_1 . Thus in the finite dimensional space of all homogeneous polynomials of degree δ in the n variables x_1, \dots, x_n , there is, up to a constant multiple, only one high weight vector, namely x_1^δ . Consequently, this space gives an irreducible representation of $\mathfrak{sl}(n, \mathbb{C})$.

3.2. Real representations of split forms of complex semisimple Lie algebras.

Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ be a semisimple Lie algebra which has a Cartan subalgebra \mathfrak{c} that is real diagonalizable. The decomposition of real representations can be obtained directly without invoking the complexification of the representation, and the algorithm for doing this is identical to the algorithm given in Section 3.1.

In more detail, for any finite dimensional real representation of \mathfrak{g} , the eigenvalues of \mathfrak{c} are all real. Thus all the roots are real valued on \mathfrak{c} . In this case, if we take a Borel subalgebra \mathfrak{b} of $\mathfrak{g}^{\mathbb{C}}$ which contains the complexification $\mathfrak{c}^{\mathbb{C}}$ as well as a positive system of roots determined by \mathfrak{b} , we have the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{c}^{\mathbb{C}} + \eta + \eta^-,$$

where \mathfrak{h} is the nil-radical of \mathfrak{b} and \mathfrak{h}^- is the nil-radical of the opposite Borel subalgebra; this \mathfrak{h}^- is generated by the root vectors corresponding to negative roots. Both \mathfrak{h} and \mathfrak{h}^- are conjugation invariant. This gives the decomposition

$$\mathfrak{g} = \mathfrak{c} + (\mathfrak{h})_\sigma + (\mathfrak{h}^-)_\sigma.$$

Thus working with $(\mathfrak{b})_\sigma$ and $(\mathfrak{h})_\sigma$ and writing this, for notational convenience, as \mathfrak{b} and \mathfrak{h} , we have $\mathfrak{b} = \mathfrak{c} + \mathfrak{h}$ and the real analogue of Lie's theorem on solvable algebras being triangularisable holds. As every \mathfrak{g} invariant subspace has a \mathfrak{g} -invariant direct summand, this means that we can work directly with \mathfrak{g} without going to its complexification and repeat the arguments of Section 2.1 to obtain an identical algorithm – namely:

Algorithm for computing irreducible components : Compute the null space of \mathfrak{h} in V . As the Cartan algebra \mathfrak{c} normalizes \mathfrak{h} , the null space of \mathfrak{h} in V decomposes into a direct sum of common eigen-spaces for \mathfrak{c} . All the weights arising in this manner are high weights and a basis of weight vectors in the null space of \mathfrak{h} in V gives the decomposition of V into its irreducible components.

The irreducible component with highest weight vector v is $\langle G_\rho v \rangle$, the span of $G_\rho v$.

Example: We want to determine the high weights in $V \otimes V$, where V is the representation of $\mathfrak{sl}(n, \mathbb{R})$ in linear polynomials in variables x_1, \dots, x_n . As $\mathfrak{sl}(n, \mathbb{R})$ is generated by copies of $\mathfrak{sl}(2, \mathbb{R})$ on the main diagonal, it is generated by the obvious copies of $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{sl}(n-1, \mathbb{R})$. This allows for inductive determination of high weight vectors.

Each element A of $\mathfrak{sl}(n, \mathbb{R})$ defines a vector field V_A on \mathbb{R}^n , namely $V_A(p) = Ap$, where p is a column vector. We identify V_A with the operation of taking the directional derivative in the direction of V_A . This gives the identification of V_A with the differential operator

$$\sum_{i,j=1}^n a_{ij} x_j \frac{\partial}{\partial x_i}.$$

The operators differential $x_i \frac{\partial}{\partial x_{i+1}}$, $1 \leq i \leq n-1$, generate a maximal ad nilpotent subalgebra $\mathfrak{sl}(n, \mathbb{R})$: denote it by \mathfrak{n} . This algebra is normalized by the maximal torus \mathfrak{c} which is generated by $x_i \frac{\partial}{\partial x_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}}$, $1 \leq i \leq n-1$.

A basis of $V \otimes V$ is $x_i \otimes x_j$, $1 \leq i, j \leq n$. Thus a basis is $x_1 \otimes x_j$, $1 \leq j \leq n$, $x_i \otimes x_1$, $2 \leq i \leq n$, and $x_k \otimes x_l$, $2 \leq k, l \leq n$. The high weights in the representation $W \otimes W$, where W is the representation of $\mathfrak{sl}(2, \mathbb{R})$ in linear polynomials in x_1, x_2 , are $x_1 \otimes x_1$ and $x_1 \otimes x_2 - x_2 \otimes x_1$.

Therefore, we may suppose inductively that the tensors in $V \otimes V$ with support in x_2, \dots, x_n , which are annihilated by $x_i \frac{\partial}{\partial x_{i+1}}$, $2 \leq i \leq n-1$, are spanned by $x_2 \otimes x_2$ and $x_2 \otimes x_3 - x_3 \otimes x_2$. Now for $2 \leq i \leq n-1$, $1 \leq j \leq n$,

$$x_i \frac{\partial}{\partial x_{i+1}} (x_1 \otimes x_j) = x_1 \otimes \delta_{j,i+1} x_i,$$

and for $2 \leq i \leq n-1$, $2 \leq j \leq n$,

$$x_i \frac{\partial}{\partial x_{i+1}} (x_j \otimes x_1) = x_i \otimes \delta_{j,i+1} x_1.$$

Therefore, the differential operators $x_i \frac{\partial}{\partial x_{i+1}}$, $2 \leq i \leq n-1$, preserve the subspace spanned by tensors involving x_1 as well as the tensors that do not involve x_1 . Consequently, if $X = \sum_{1 \leq i, j \leq n} x_i \otimes x_j$ is annihilated by $x_k \frac{\partial}{\partial x_{k+1}}$, $2 \leq k \leq n-1$, then it is a linear combination of terms involving x_1 , $x_2 \otimes x_2$ and $x_2 \otimes x_3 - x_3 \otimes x_2$. Hence for $2 \leq k \leq n-1$, the equation $x_k \frac{\partial}{\partial x_{k+1}} (X) = 0$ requires

$$a_{1(k+1)} x_1 \otimes x_k + a_{(k+1)1} x_k \otimes x_1 = 0.$$

Therefore $a_{1(k+1)} = 0$, $a_{(k+1)1} = 0$ for $2 \leq k \leq n-1$.

Thus $X = a_{11} x_1 \otimes x_1 + a_{12} x_1 \otimes x_2 + a_{21} x_2 \otimes x_1 + a_{22} x_2 \otimes x_2 + a_{23} (x_2 \otimes x_3 - x_3 \otimes x_2)$.

As X must be annihilated also by $x_1 \frac{\partial}{\partial x_2}$, it now follows that the linearly independent high weight vectors in $V \otimes V$ are $x_1 \otimes x_1$ and $x_1 \otimes x_2 - x_2 \otimes x_1$.

4. REAL REPRESENTATIONS OF REAL SEMISIMPLE LIE ALGEBRAS

In this section, we want to describe algorithmically the high weight vectors in the complexification of a given real representation V of a real semisimple Lie algebra \mathfrak{g} and then decompose $V^{\mathbb{C}}$ into conjugation invariant and \mathfrak{g} -invariant subspaces that are either irreducible or sums of an irreducible subspace and its conjugate. Taking the real points then gives the decomposition of V into its \mathfrak{g} -invariant subspaces. One of the main technical points is the construction of an element of the Weyl group that maps a suitably chosen Borel algebra to its conjugate. We first need to recall some standard facts about finite dimensional representations of real semisimple Lie algebras. For the convenience of the reader, proofs of some standard results are also outlined briefly.

Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ be a real semisimple Lie algebra, and let ρ be a representation of \mathfrak{g} in $\mathfrak{gl}(V)$, where V is a finite dimensional real vector space. Let G_ρ be the subgroup of $GL(V)$ whose Lie algebra is $\rho(\mathfrak{g})$ and $G_\rho^{\mathbb{C}}$ the subgroup of $GL(V^{\mathbb{C}})$ whose Lie algebra is $\rho(\mathfrak{g}) + \sqrt{-1} \cdot \rho(\mathfrak{g})$. The group $G_\rho^{\mathbb{C}}$ is an algebraic subgroup of $GL(V^{\mathbb{C}})$. This group is generated by the subgroups $\{\exp zX \mid z \in \mathbb{C}\}$, $X \in \mathfrak{g}$, and it is an image of the simply connected complex group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$. We denote this simply connected group by $G_{sc}^{\mathbb{C}}(\mathfrak{g})$. The group G_ρ is the connected component of the locus of real points of $G_\rho^{\mathbb{C}}$.

We first prove a version of Weyl's theorem on complete reducibility of representations in the following form:

Proposition 4.1. *Every \mathfrak{g} -invariant subspace of V has a \mathfrak{g} -invariant complementary summand.*

Proof. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of \mathfrak{g} . The algebra $\mathfrak{k} + \sqrt{-1} \cdot \mathfrak{p}$ is compact and so is its image $\rho(\mathfrak{k}) + \sqrt{-1} \cdot \rho(\mathfrak{p})$. Let K be the subgroup of $G_\rho^{\mathbb{C}}$ corresponding to

$\rho(\mathfrak{k}) + \sqrt{-1} \cdot \rho(\mathfrak{p})$. This group K is compact and is invariant under the conjugation σ on $V^{\mathbb{C}}$ defined by V . Moreover, its complexification is $G_{\rho}^{\mathbb{C}}$.

Take a Hermitian inner product H on $V^{\mathbb{C}}$ which is invariant under K . Then $\tilde{H} = H + \sigma(H)$ is a Hermitian inner product that is both K and σ -invariant.

Let U be a subspace of V which is G_{ρ} -invariant. Then its complexification $U^{\mathbb{C}}$ is conjugation as well as G_{ρ} invariant. Thus it is $G_{\rho}^{\mathbb{C}} = K^{\mathbb{C}}$ invariant. Consequently, its orthogonal complement is conjugation and K and hence $K^{\mathbb{C}}$ invariant. But then it is $G_{\rho}^{\mathbb{C}}$ and therefore G_{ρ} invariant. Hence $V^{\mathbb{C}} = U^{\mathbb{C}} + (U^{\mathbb{C}})^{\perp}$ gives the decomposition $V = U + (U^{\mathbb{C}})^{\perp}_{\sigma}$ into G_{ρ} -invariant summands. \square

Fix a Cartan subalgebra \mathfrak{c} of a semisimple Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$. The algebra $\mathfrak{c}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. A positive system of roots — for example one which is determined by fixing a basis of \mathfrak{c} as explained in Section 2.1 — determines a Borel subalgebra \mathfrak{b} of $\mathfrak{g}^{\mathbb{C}}$ which contains $\mathfrak{c}^{\mathbb{C}}$. We have the decomposition $\mathfrak{b} = \mathfrak{c}^{\mathbb{C}} + \mathfrak{h}$, where \mathfrak{h} is the nil-radical of \mathfrak{b} . Now conjugation σ maps $\mathfrak{c}^{\mathbb{C}}$ to itself, therefore it permutes all the roots of $\mathfrak{c}^{\mathbb{C}}$.

In general, if λ is a complex linear function on $\mathfrak{c}^{\mathbb{C}}$, then the complex linear function λ^{σ} is defined by $\lambda^{\sigma}(H) = \overline{\lambda(\sigma(H))}$, where $H \in \mathfrak{c}^{\mathbb{C}}$.

Let R be the set of roots of $\mathfrak{c}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$ and R^+ the positive system of roots determined by \mathfrak{b} . For a root α , let $(\mathfrak{g}^{\mathbb{C}})_{\alpha}$ be the corresponding root space. Thus as

$$\mathfrak{h} = \sum_{\alpha > 0} (\mathfrak{g}^{\mathbb{C}})_{\alpha},$$

the conjugate $\sigma(\mathfrak{h})$ is the subalgebra $\sum_{\alpha > 0} (\mathfrak{g}^{\mathbb{C}})_{\alpha}^{\sigma}$.

Lemma 4.2. *Let $\rho(\alpha) = \alpha^{\sigma}$. For a simple root α , let w_{α} be the corresponding involution in the Weyl group of $\mathfrak{c}^{\mathbb{C}}$. Then there are simple roots β_1, \dots, β_l which are determined algorithmically such that $p(w_{\beta_1} \dots w_{\beta_l}(R^+)) = R^+$ and $w_{\beta_1} \dots w_{\beta_l}(R^+) = p(R^+)$, where p is the permutation of the roots given by complex conjugation σ . (Cf. [Iw, Lemma 9].)*

Proof. If $\sigma(\mathfrak{h})$ is not equal to \mathfrak{h} , there must be a simple positive root α such that α^{σ} is negative.

List the positive roots which are mapped to negative roots by p as $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_k$ and let $\alpha_{k+1}, \dots, \alpha_N$ be the remaining positive roots, so that $p(\alpha_{k+1}), \dots, p(\alpha_N)$ are all positive.

Now w_{α_1} maps α_1 to $-\alpha_1$ and permutes the remaining positive roots, that is

$$w_{\alpha_1}(R^+) = \{-\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_N\}.$$

Therefore

$$p(w_{\alpha_1}(R^+)) = \{-p(\alpha_1), p(\alpha_2), \dots, p(\alpha_k), p(\alpha_{k+1}), \dots, p(\alpha_N)\}.$$

In this list, only $p(\alpha_2), \dots, p(\alpha_k)$ are negative. Thus the number of positive roots mapped to negative roots by pw_{α_1} decreases by 1.

Replacing p by $p \circ w_{\alpha_1}$ and repeating the argument we see that there are simple roots $\beta_1 = \alpha_1, \dots, \beta_l$ such that $p(w_{\beta_1} \dots w_{\beta_l}(R^+)) = R^+$, and therefore $w_{\beta_1} \dots w_{\beta_l}(R^+) = p(R^+)$ as p is an involution. \square

Corollary 4.3. *Using the notation of Section 2, there is an element ω constructed algorithmically, namely $\omega = w_{\beta_1} \dots w_{\beta_l}$, which lies in the normalizer of $\mathfrak{c}^{\mathbb{C}}$ in the group $G^{\mathbb{C}}$ such that $\omega \eta \omega^{-1} = \sigma(\eta)$, where σ denotes complex conjugation in $\text{GL}(n, \mathbb{C})$.*

4.1. Correspondence between the weight of a high weight vector and its conjugate in a representation. To determine the correspondence between the weight of a high weight vector and its conjugate in a representation, we need to recall some basic facts on the Weyl group [St1, p. 27, Lemma 19]. The Weyl group has representatives in the group $G^{\mathbb{C}}$ which are given by the following construction.

For each simple root α take a standard copy of $\mathfrak{sl}(2, \mathbb{C})$ with generators $X_{\alpha}, X_{-\alpha}$ and commutation relations

$$[X_{\alpha}, X_{-\alpha}] = H_{\alpha}, [H_{\alpha}, X_{\alpha}] = 2X_{\alpha}, [H_{\alpha}, X_{-\alpha}] = -2X_{-\alpha}.$$

Let $w_{\alpha} = \exp(X_{\alpha}) \exp(-X_{-\alpha}) \exp(X_{\alpha})$. Then w_{α} represents reflection along the root α [St1, p. 27, Lemma 19].

Although a representation ρ of \mathfrak{g} in $\mathfrak{gl}(V)$, where V is a finite dimensional real vector space, does not necessarily give a representation of the group G_{ρ} , the element

$$\rho^{\mathbb{C}}(\exp(X)Y \exp(-X)),$$

where $X, Y \in \mathfrak{g}^{\mathbb{C}}$, can be easily calculated.

We have $\exp(X)Y \exp(-X) = \exp(\text{ad}(X))(Y)$; therefore

$$\rho^{\mathbb{C}}(e^X Y e^{-X}) = e^{\text{ad} \rho^{\mathbb{C}}(X)} \rho^{\mathbb{C}}(Y) = e^{\rho^{\mathbb{C}}(X)} \rho^{\mathbb{C}}(Y) e^{-\rho^{\mathbb{C}}(X)}$$

(exponential notation is switched for convenience). This means that if we set $w_{\alpha, \rho} = e^{\rho(X_{\alpha})} e^{-\rho(X_{-\alpha})} e^{\rho(X_{\alpha})}$, then

$$\rho^{\mathbb{C}}(w_{\alpha} Y w_{\alpha}^{-1}) = w_{\alpha, \rho} \rho^{\mathbb{C}}(Y) w_{\alpha, \rho}^{-1}.$$

Hence if we write the element ω , in the notation of Corollary 4.3, as $\omega = w_{\beta_1} \dots w_{\beta_l}$, where the w_{β_j} , $j = 1, \dots, l$, are constructed as in Lemma 4.2, then

$$\rho^{\mathbb{C}}(w_{\beta_1} \dots w_{\beta_l} Y (w_{\beta_1} \dots w_{\beta_l})^{-1}) = w_{\beta_1, \rho} \dots w_{\beta_l, \rho} \rho^{\mathbb{C}}(Y) (w_{\beta_1, \rho} \dots w_{\beta_l, \rho})^{-1}.$$

Notation : $\omega_{\rho} = w_{\beta_1, \rho} \dots w_{\beta_l, \rho}$, where ω is defined as in Corollary 4.3.

Recall the context: \mathfrak{c} is a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$. Thus $\mathfrak{c}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. A positive system of roots, for example one which is determined by fixing a basis of \mathfrak{c} as explained in Section 2.1, determines a Borel subalgebra \mathfrak{b} of $\mathfrak{g}^{\mathbb{C}}$ which contains $\mathfrak{c}^{\mathbb{C}}$. We have the decomposition $\mathfrak{b} = \mathfrak{c}^{\mathbb{C}} + \mathfrak{h}$, where \mathfrak{h} is the nil-radical of \mathfrak{b} .

Let R be the set of roots of $\mathfrak{c}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$ and R^+ the positive system of roots determined by \mathfrak{b} . For a root α , let $(\mathfrak{g}^{\mathbb{C}})_{\alpha}$ be the corresponding root space. Thus $\mathfrak{h} = \sum_{\alpha > 0} (\mathfrak{g}^{\mathbb{C}})_{\alpha}$, and the conjugate $\sigma(\mathfrak{h}) = \bar{\mathfrak{h}}$ is the subalgebra $\sum_{\alpha > 0} (\mathfrak{g}^{\mathbb{C}})_{\alpha}^{\sigma}$.

Lemma 4.4. *Let v be a high weight vector relative to the pair $(\mathfrak{h}, \mathfrak{c}^{\mathbb{C}})$ in the complexification $\rho^{\mathbb{C}}$ of a finite dimensional representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ with weight λ . Then $\omega_{\rho}^{-1}\bar{v}$ is a high weight vector for the pair $(\mathfrak{h}, \mathfrak{c}^{\mathbb{C}})$ with weight $\omega^{-1}\lambda^{\sigma}$.*

Moreover, the irreducible component containing $\omega_{\rho}^{-1}\bar{v}$ is the linear span $\langle G_{\rho}^{\mathbb{C}}\bar{v} \rangle$, where $G_{\rho}^{\mathbb{C}}$ is the subgroup of $\mathrm{GL}(V^{\mathbb{C}})$ generated by the complex one parameter subgroups $\{\exp(z\rho(X)) \mid z \in \mathbb{C}\}$, $X \in \mathfrak{g}$. (Cf. [Iw, Theorem 2].)

Proof. By definition $\rho^{\mathbb{C}}(\mathfrak{h}).v = 0$ and therefore $\rho^{\mathbb{C}}(\bar{\mathfrak{h}}).\bar{v} = 0$. Hence by Corollary 4.3,

$$\rho^{\mathbb{C}}(\omega\mathfrak{h}\omega^{-1}).\bar{v} = 0.$$

Using the identity $\rho^{\mathbb{C}}(\omega Y \omega^{-1}) = \omega_{\rho} \rho^{\mathbb{C}}(Y) \omega_{\rho}^{-1}$, we see that

$$\rho^{\mathbb{C}}(\mathfrak{h})\omega_{\rho}^{-1}\bar{v} = 0.$$

Now, for all $H \in \mathfrak{c}^{\mathbb{C}}$,

$$\rho(H)v = \lambda(H)v$$

implies that $\rho(H)\bar{v} = \lambda^{\sigma}(H)\bar{v}$. Therefore,

$$\begin{aligned} \rho(H)\omega_{\rho}^{-1}\bar{v} &= \omega_{\rho}^{-1}\omega_{\rho}\rho(H)\omega_{\rho}^{-1}\bar{v} = \omega_{\rho}^{-1}\rho(\omega H \omega^{-1})\bar{v} \\ &= \omega_{\rho}^{-1}\lambda^{\sigma}(\omega H \omega^{-1})\bar{v} = \lambda^{\sigma}(\omega H \omega^{-1})(\omega_{\rho}^{-1}\bar{v}) = (\omega^{-1}\lambda^{\sigma}(H))(\omega_{\rho}^{-1}\bar{v}). \end{aligned}$$

The proof of the last statement of the lemma is clear. \square

Notation : For a complex linear function λ on $\mathfrak{c}^{\mathbb{C}}$, we will write $\Theta(\lambda) = \omega^{-1}\lambda^{\sigma}$.

Lemma 4.5. *The above function Θ is an involution.*

Proof. For any $n \in N_{G^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}})$, we have $(n.\lambda^{\sigma})^{\sigma} = \bar{n}\lambda$. Now using Corollary 4.3 and the notation introduced there,

$$\omega\mathfrak{h}\omega^{-1} = \bar{\mathfrak{h}}.$$

Consequently, $\bar{\omega}\bar{\mathfrak{h}}\bar{\omega}^{-1} = \mathfrak{h}$ and

$$\bar{\omega}\omega\mathfrak{h}\omega^{-1}\bar{\omega}^{-1} = \mathfrak{h}.$$

Hence $\bar{\omega}\omega$ is an element of the Weyl group that maps every positive root to a positive root. Thus $(\bar{\omega}\omega).\lambda = \lambda$. A computation then shows that $\Theta^2(\lambda) = \lambda$. \square

We will apply these lemmas to decompose a given real representation V into its real irreducible components.

Now if we have a representation ρ of \mathfrak{g} in $\mathfrak{gl}(V)$, where V is a finite dimensional real vector space, then its complexification $\rho^{\mathbb{C}}$ is a representation of $\mathfrak{g}^{\mathbb{C}}$ in $\mathfrak{gl}(V^{\mathbb{C}})$. As in Section 2, compute the null space of \mathfrak{h} in $V^{\mathbb{C}}$, and the weights of $\mathfrak{c}^{\mathbb{C}}$ in this space determine the high weights in the representation in $\mathfrak{gl}(V^{\mathbb{C}})$. Denote this set of high weights by $\mathcal{P}(V^{\mathbb{C}})$.

For $\lambda \in \mathcal{P}(V^{\mathbb{C}})$, let $V^{\mathbb{C}}(\lambda)$ be the isotypical component with weight λ . It is the $\mathfrak{g}^{\mathbb{C}}$ -invariant subspace of $V^{\mathbb{C}}$ generated by high weight vectors of weight λ . Computing a basis of eigenvectors of $\mathfrak{c}^{\mathbb{C}}$ in the null space of \mathfrak{h} for the weight λ determines the isotypical component $V^{\mathbb{C}}(\lambda)$.

The involution Θ operates on $\mathcal{P}(V^{\mathbb{C}})$. If the high weights λ and $\Theta(\lambda)$ are distinct, then $V^{\mathbb{C}}(\lambda) + V^{\mathbb{C}}(\Theta(\lambda))$ is a direct sum. Moreover, by Corollary 4.3,

$$V^{\mathbb{C}}(\Theta(\lambda)) = \overline{V^{\mathbb{C}}(\lambda)}.$$

Lemma 4.6. *If M and \overline{M} are irreducible $\mathfrak{g}^{\mathbb{C}}$ -invariant subspaces in $V^{\mathbb{C}}$ with different high weights relative to the pair $(\mathfrak{h}, \mathfrak{c}^{\mathbb{C}})$, then the real points in the direct sum $M + \overline{M}$ give an irreducible \mathfrak{g} -invariant subspace.*

Proof. Let λ be the high weight of M . Then $\Theta(\lambda)$ is the high weight of \overline{M} , and by assumption λ and $\Theta(\lambda)$ are distinct. Thus the sum $M + \overline{M}$ has only these two high weights. Let S be a \mathfrak{g} -irreducible subspace of $(M + \overline{M})_{\sigma}$. Now $S^{\mathbb{C}}$ contains a $\mathfrak{g}^{\mathbb{C}}$ invariant irreducible subspace of $M + \overline{M}$, which must be either M or \overline{M} . As $S^{\mathbb{C}}$ is conjugation invariant, it must therefore contain both M and \overline{M} . Hence $S^{\mathbb{C}} = M + \overline{M}$ and hence $S = (M + \overline{M})_{\sigma}$. \square

For the following Lemma 4.7, we are assuming that the null space of \mathfrak{h} in $V^{\mathbb{C}}$ has been computed and a basis of the eigen-spaces of $\mathfrak{c}^{\mathbb{C}}$ in this null space is known explicitly.

Lemma 4.7. *If M is an irreducible $\mathfrak{g}^{\mathbb{C}}$ -invariant subspaces in $V^{\mathbb{C}}$ with highest weight vector v , then $M \cap \overline{M} \neq 0$ if and only if $v \wedge \omega_{\rho}^{-1}\overline{v} = 0$ in which case $M = \overline{M}$ and M_{σ} is a \mathfrak{g} -invariant irreducible subspace of V . Moreover, in this case*

$$M = \langle G_{\rho}^{\mathbb{C}}v \rangle = \langle G_{\rho}^{\mathbb{C}}\overline{v} \rangle = \langle G_{\rho}^{\mathbb{C}}\omega^{-1}\overline{v} \rangle.$$

Proof. Clearly if M is irreducible and conjugation invariant, then its real points give an irreducible \mathfrak{g} -invariant subspace of V . The remaining assertions follow from Lemma 4.4. \square

Lemma 4.8 also follows from Lemma 7.9 of [KS].

Lemma 4.8. *If M is an irreducible $\mathfrak{g}^{\mathbb{C}}$ -invariant subspaces in $V^{\mathbb{C}}$ with highest weight vector v , then $M \cap \overline{M} = 0$ if and only if $v \wedge \omega_{\rho}^{-1}\overline{v} \neq 0$. Moreover, if $v \wedge \omega_{\rho}^{-1}\overline{v} \neq 0$ and highest weight λ of M and the highest weight $\Theta(\lambda)$ of \overline{M} are equal, then $M + \overline{M}$ is a direct sum. Denoting the real points of $M \oplus \overline{M}$ by W , the \mathfrak{g} -module W is either irreducible or it is of the form $B \oplus B$, with B irreducible. The map $w_{\rho}^{-1}\overline{w_{\rho}^{-1}}$ of M is a multiple of the identity map by a nonzero real number d . If this scalar d is positive, then W is the G_{ρ} span of the real (respectively, imaginary) part of $v_1 = v + \sqrt{d}w_{\rho}^{-1}\overline{v}$ (respectively, $v_2 = v - \sqrt{d}w_{\rho}^{-1}\overline{v}$).*

If this scalar d is negative, then W is irreducible.

Proof. The first statement is a consequence of Lemma 4.4.

Suppose that W has more than one component. Let

$$W = W_1 + \dots + W_k$$

be the irreducible components with $k \geq 2$. Each of $W_i^{\mathbb{C}} = W_i \otimes_{\mathbb{R}} \mathbb{C}$, $1 \leq i \leq k$, has a $\mathfrak{g}^{\mathbb{C}}$ -invariant irreducible subspace V_i with the same highest weight λ . As the dimension of any irreducible subspace is determined by the highest weight, we see that $k = 2$, and both $W_1^{\mathbb{C}}$ and $W_2^{\mathbb{C}}$ are irreducible. Moreover, $W_1^{\mathbb{C}}$ and $W_2^{\mathbb{C}}$ are both self-conjugate, and share a common highest weight. Thus their high weight vectors must be linear combinations of v and $w_{\rho}^{-1}\bar{v}$. Therefore, we need to determine when a subspace with such a high weight vector is self-conjugate.

Let the highest weight of $W_1^{\mathbb{C}}$ be $v + \mu w_{\rho}^{-1}\bar{v}$, with $\mu \neq 0$. Then

$$W_1^{\mathbb{C}} = \langle G_{\rho}^{\mathbb{C}}(v + \mu w_{\rho}^{-1}\bar{v}) \rangle.$$

The conjugate of $W_1^{\mathbb{C}}$ is

$$\langle G_{\rho}^{\mathbb{C}}(\bar{v} + \bar{\mu} w_{\rho}^{-1}v) \rangle = \langle G_{\rho}^{\mathbb{C}}(w_{\rho}^{-1}\bar{v} + \bar{\mu} w_{\rho}^{-1} w_{\rho}^{-1}v) \rangle.$$

Consider the composition of $\mathfrak{g}^{\mathbb{C}}$ -linear maps $M \longrightarrow \bar{M} \longrightarrow M$ determined by the mapping of high weights

$$v \longmapsto w_{\rho}^{-1}\bar{v} \longmapsto w_{\rho}^{-1} \overline{w_{\rho}^{-1}\bar{v}} = w_{\rho}^{-1} w_{\rho}^{-1}v.$$

By Schur's lemma, $w_{\rho}^{-1} \overline{w_{\rho}^{-1}}$ must be a nonzero complex number, say d . Now note that this scalar d must be a real number because w_{ρ}^{-1} and $\overline{w_{\rho}^{-1}}$ must commute with each other. Therefore, the self-conjugacy condition of $W_1^{\mathbb{C}}$ requires that

$$(v + \mu \cdot w_{\rho}^{-1}\bar{v}) \wedge (w_{\rho}^{-1}\bar{v} + \bar{\mu} \cdot w_{\rho}^{-1} \overline{w_{\rho}^{-1}\bar{v}}) = (v + \mu \cdot w_{\rho}^{-1}\bar{v}) \wedge (w_{\rho}^{-1}\bar{v} + \bar{\mu}d \cdot v) = 0.$$

This means that

$$v \wedge w_{\rho}^{-1}\bar{v} + |\mu|^2 d \cdot (w_{\rho}^{-1}\bar{v}) \wedge v = 0.$$

As $v \wedge w_{\rho}^{-1}\bar{v} \neq 0$, this implies that

$$|\mu|^2 d = 1.$$

Therefore, d must be a positive real number.

If this scalar d is negative, there are no self-conjugate invariant subspaces of $M + \bar{M}$, under the assumptions that

- $M \cap \bar{M} = 0$, and
- both M and \bar{M} have the same highest weight.

In case d is a positive real number, we can take $W_1^{\mathbb{C}}$ and $W_2^{\mathbb{C}}$ to be the $G_{\rho}^{\mathbb{C}}$ invariant subspaces with highest weights $v_1 = v + \sqrt{d}w_{\rho}^{-1}\bar{v}$ and $v_2 = v - \sqrt{d}w_{\rho}^{-1}\bar{v}$. If this scalar d is a positive real number, then the real points in $M + \bar{M}$ are the G_{ρ} spans of the real (or imaginary) parts of $v_1 = v + \sqrt{d}w_{\rho}^{-1}\bar{v}$ and $v_2 = v - \sqrt{d}w_{\rho}^{-1}\bar{v}$ respectively. \square

Lemma 4.9. *Let $V^{\mathbb{C}}(\lambda)$ be the isotypical component of weight λ . If $\lambda = \omega^{-1}\lambda^\sigma$, then any $\mathfrak{g}^{\mathbb{C}}$ -invariant and conjugation invariant proper subspace M of $V^{\mathbb{C}}(\lambda)$ has a $\mathfrak{g}^{\mathbb{C}}$ and conjugation invariant summand.*

Let w_1, \dots, w_d be a basis of high weight vectors of M . Then there is a high weight vector v such that

- *either $w_1 \wedge \dots \wedge w_d \wedge v \neq 0$ and $v \wedge \omega^{-1}\bar{v} = 0$,*
- *or $v \wedge \omega^{-1}\bar{v} \neq 0$ and $w_1 \wedge \dots \wedge w_d \wedge v \wedge \omega^{-1}\bar{v} \neq 0$.*

Proof. This is an immediate consequence of Proposition 4.1 and Lemma 4.4, taking into account that $\lambda = \omega^{-1}\lambda^\sigma$ implies that $V^{\mathbb{C}}(\lambda)$ is self-conjugate. \square

Remark 4.10. If $\lambda = \omega^{-1}\lambda^\sigma$ and v_1, \dots, v_l is a basis of high weight vectors in the isotypical component $V^{\mathbb{C}}(\lambda)$, then starting with v_1 and applying Lemma 4.9, we can decompose $V^{\mathbb{C}}(\lambda)$ into $\mathfrak{g}^{\mathbb{C}}$ -invariant subspaces which are either irreducible or a sum of two irreducible components.

We can now formulate a computational procedure for decomposing a given representation ρ of \mathfrak{g} in a real finite dimensional space V into \mathfrak{g} -invariant subspaces which are either irreducible or real points of a self-conjugate subspace of $V^{\mathbb{C}}$ that is a sum of two irreducible $\mathfrak{g}^{\mathbb{C}}$ -invariant subspaces with the same highest weight. The real points of this subspace are described explicitly in Lemma 4.8.

Step 1 : Choose a Cartan subalgebra \mathfrak{c} of \mathfrak{g} and compute its roots in $\mathfrak{g}^{\mathbb{C}}$ — and a positive system of roots — for example by fixing a basis of \mathfrak{c} (cf. Section 2). The positive roots then determine a Borel subalgebra \mathfrak{b} of $\mathfrak{g}^{\mathbb{C}}$ with $\mathfrak{b} = \mathfrak{c} + \mathfrak{n}$, where \mathfrak{n} is the nil-radical of \mathfrak{b} .

Step 2 : Determine a basis of high weight vectors and the corresponding weights in $V^{\mathbb{C}}$ using the procedure given in Section 3: namely, compute the root vectors in \mathfrak{n} corresponding to the simple roots and compute their joint invariants. This amounts to solving linear equations. In more detail, one lists all the simple roots and the corresponding root vectors. Using the Jordan form for the first root vector, to facilitate the computations, one finds its invariants and then works systematically with the remaining root vectors corresponding to simple roots and the Jordan forms of the root vectors to find all the joint invariants and a basis thereof. As $\mathfrak{c}^{\mathbb{C}}$ normalizes \mathfrak{n} , it operates on the joint invariants of \mathfrak{n} . A basis for the corresponding eigen-spaces gives all the high weight vectors.

The algorithm for decomposing complex representations is implemented in SLA [deG1].

If one is working with spaces of functions, finding joint invariants amounts to solving a system of linear partial differential equations. One can use row reductions for the operators corresponding to simple roots [ABGM], or the lower central series of the nil-radical and Jordan forms to systematically solve the system of partial differential equations.

Step 3 : Compute the element ω — using Corollary 4.3 — to decide self-conjugacy of weights. For self-conjugate weights, with more than one linearly independent weight vector, compute $\omega_\rho^{-1}\overline{\omega_\rho^{-1}}v$ for such a high weight vector v , where ω_ρ is defined as in Section 4.1.

Step 4 : Compute the orbits of the involution Θ defined by $\Theta(\lambda) = \omega^{-1}\lambda^\sigma$, where λ is a high weight in $V^{\mathbb{C}}$, given in Step 2.

Step 5 : Choose representatives $\lambda_1, \dots, \lambda_k$ of orbits of length two and $\lambda_{k+1}, \dots, \lambda_r$ representatives of orbits of length one.

Step 6 : If λ is a representative of one of the orbits of length two, choose a basis of high weight vectors in the isotypical component $V^{\mathbb{C}}(\lambda)$, using Step 2. If v is such a high weight vector, then the real points in $\langle G_\rho^{\mathbb{C}}v \rangle + \langle G_\rho^{\mathbb{C}}\bar{v} \rangle$ give an irreducible \mathfrak{g} -invariant summand (Lemma 4.6). In this case this summand is $\langle G_\rho(v + \bar{v}) \rangle$.

Step 7 : If λ is a representative of one of the orbits of length one, decompose the isotypical component $V^{\mathbb{C}}(\lambda)$ into $\mathfrak{g}^{\mathbb{C}}$ -invariant and conjugation invariant subspaces which are either irreducible or a sum of two irreducible components of $M \oplus \overline{M}$ — using Lemma 4.9.

Denoting the real points of $M \oplus \overline{M}$ by W , the \mathfrak{g} -module W is either irreducible or it is of the form $B \oplus B$, with B irreducible. Describe the real points by using Lemma 4.8.

5. FURTHER EXAMPLES

In this section, in contrast to the general examples given in Section 3, we consider in detail representations of real Lie algebras of type $A_1 \times A_1$ — because of their importance in Physics. A certain amount of repetition is inevitable as these algebras complexify to the same algebra. However, for the convenience of the reader and for readability, detailed information for each of these algebras has been given separately.

Before working out specific examples, a word about normalizations of structure constants is in order.

Any torus of a real Lie algebra L has a compact part and a split part. In any representation, the eigenvalues of the compact part are purely imaginary and the eigenvalues of the split part are real. For this reason, as a check on computations, the generators in the complexification of L have at times not been normalized to give the standard $\mathfrak{sl}(2)$ relations for the root $\mathfrak{sl}(2)$ corresponding to simple roots.

In computations of high weights, one needs to find the joint invariants of the nil-radical of a Borel subalgebra that contains the complexification of a real Cartan subalgebra. As far as vector spaces of differentiable functions are concerned, one can use row reductions

as in [ABGM] for the vector fields corresponding to simple roots to determine the number of functionally independent invariants. When working with representations in vector spaces of polynomials, one looks for invariants in linear polynomials, then quadratic polynomials and so on till the required number of functionally independent invariants has been obtained. This gives a certain number of high weights. We have used the following fact to verify that the high weights have been computed correctly: if X_1, \dots, X_N are the root vectors corresponding to all the positive roots and Y_1, \dots, Y_N are the root vectors corresponding to all the negative roots and v is a high weight vector, then $Y_N^{d_N} \dots Y_1^{d_1} v$, where the d_i vary over all nonnegative integers, span the corresponding irreducible space: as all the operators Y_1, \dots, Y_N are nilpotent, the integers d_i vary over a finite set. This can also be verified by using Weyl's dimension formula. As far as representations in spaces of homogeneous polynomials of a given degree are concerned, one can do this by software directly and this amounts to solving linear equations.

In this section we first decompose $\text{End}(\mathbb{R}^3)$ under the left action of $\text{SO}(3)$ and then we give examples of representations of the algebras $\mathfrak{so}(3)$, $\mathfrak{so}(4)$, $\mathfrak{so}(1,3)$ and $\mathfrak{so}(2,2)$ in the space of homogeneous polynomials of degree d and decompose them into their real irreducible components, using the procedure given in Section 4. The representations come from linear vector fields, and the Jordan form of nilpotent matrices is helpful in computing the high weights. We will use the following change of variables formulae.

If $A = (a_{ij})$ is an $n \times n$ matrix, then it defines a vector field V_A , where $V_A(p) = Ap$: here p is a point in \mathbb{C}^n . As a differential operator, we have $V_A = (\partial_{x_1}, \dots, \partial_{x_n}) A(x_1, \dots, x_n)^t$, where ∂_{x_i} denotes the partial derivative with respect to variable x_i . Hence, if $C^{-1}AC = D$ and we make a change of coordinates $(\tilde{x}_1, \dots, \tilde{x}_n)^t = C^{-1}(x_1, \dots, x_n)^t$, then $(\partial_{x_1}, \dots, \partial_{x_n}) A(x_1, \dots, x_n)^t = (\partial_{\tilde{x}_1}, \dots, \partial_{\tilde{x}_n}) D(\tilde{x}_1, \dots, \tilde{x}_n)^t$.

Recall that if A is an $n \times n$ diagonal matrix with diagonal entries 1 and -1 on the main diagonal then the Lie algebra of the corresponding orthogonal group has generators $e_{ij} - e_{ji}$ if $a_{ij}a_{ji} = 1$, and $e_{ij} + e_{ji}$ if $a_{ij}a_{ji} = -1$. Moreover, if A has the first p diagonal entries 1, and the next q diagonal entries -1 , where $p + q = n$, then the Lie algebra of the corresponding orthogonal group is generated by

$$e_{12} - e_{21}, \dots, e_{p-1p} - e_{pp-1}, e_{pp+1} + e_{p+1p}, e_{p+1p+2} - e_{p+2p+1}, \dots, e_{n-1n} - e_{nn-1}.$$

Example 5.1. Decomposition of the representation of $\text{SO}(3, \mathbb{R})$ on 3×3 matrices with real entries under the left action of $\text{SO}(3, \mathbb{R})$.

The Lie algebra of $\mathfrak{so}(3)$ is generated by

$$I = e_{12} - e_{21}, J = e_{23} - e_{32} \text{ and } [I, J] = K = e_{13} - e_{31}.$$

As the commutation relations are

$$\begin{aligned} [I, J - \sqrt{-1}K] &= \sqrt{-1}(J - \sqrt{-1}K), [I, J + \sqrt{-1}K] = -\sqrt{-1}(J + \sqrt{-1}K), \\ [J - \sqrt{-1}K, J + \sqrt{-1}K] &= 2\sqrt{-1}I, \end{aligned}$$

we see that $\mathfrak{so}(3)^\mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$. Moreover $X = J - \sqrt{-1}K$, $Y = J + \sqrt{-1}K$ and $H = [X, Y]$ satisfy the canonical relations of $\mathfrak{sl}(2, \mathbb{C})$. We can take $\langle I \rangle$ as a Cartan subalgebra of $\mathfrak{so}(3)$. Let $\omega = e^X e^{-Y} e^X$; a computation shows that

$$\omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus we have $\omega I \omega^{-1} = -I$.

In any finite dimensional representation of $\mathfrak{so}(3)$, all the nonzero eigen values must be purely imaginary eigenvalues. Thus if we complexify a real representation of $\mathfrak{so}(3)$, and v is an eigenvector for I with eigenvalue $n\sqrt{-1}$, and we define the weight λ by $\lambda(I) = n\sqrt{-1}$, we obtain $(\omega^{-1} \lambda^\sigma)(I) = \lambda(I)$ as I is a real element. Therefore, all the weights are self-conjugate. As noted already, the positive and negative root space for $\mathfrak{so}(3)^\mathbb{C}$ are $X = J - \sqrt{-1}K$, $Y = J + \sqrt{-1}K$. Hence both X and Y must be nilpotent.

Now consider the homomorphism

$$\rho : \mathfrak{so}(3, \mathbb{R}) \longrightarrow \mathfrak{gl}(\text{End}(\mathbb{R}^3))$$

given by $\rho(A)(X) = AX$. Now $\langle I, J - \sqrt{-1}K \rangle = \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{so}(3, \mathbb{C})$. The null space of $X = J - \sqrt{-1}K$ works out to be

$$\langle e_{11} + \sqrt{-1}e_{21}, e_{12} + \sqrt{-1}e_{22}, e_{13} + \sqrt{-1}e_{23} \rangle$$

and I operates as multiplication by $\sqrt{-1}$ on this null space of X . If e is such an eigenvector, then the $\mathfrak{so}(3, \mathbb{C})$ -invariant subspace containing e is the span of $\langle e, Ye, Y^2e \rangle$ with eigenvalues $\sqrt{-1}, 0, -\sqrt{-1}$. Therefore, the irreducible components of the representation of $\mathfrak{so}(3, \mathbb{C})$ on $\mathfrak{gl}(\text{End}(\mathbb{C}^3))$ are the same as the representation of $\mathfrak{sl}(2, \mathbb{C})$ on homogeneous quadratic polynomials. These representations are self-conjugate; indeed, if e is a high weight vector then $\omega \cdot e = \bar{e}$.

Working with the real (or imaginary) parts of the high weight vectors, we get the decomposition

$$\text{End}(\mathbb{R}^3) = \langle \text{SO}(3) \cdot e_{11} \rangle \oplus \langle \text{SO}(3) \cdot e_{12} \rangle \oplus \langle \text{SO}(3) \cdot e_{13} \rangle.$$

Example 5.2. Decomposition of the representation $\mathfrak{so}(3)$ in real homogeneous polynomials of degree d in 3 real variables.

We will use the notation of Example 5.1. As noted already, $X = J - \sqrt{-1}K$ and $Y = J + \sqrt{-1}K$ give the positive and negative root space for $\mathfrak{so}(3)^\mathbb{C}$. Therefore both X and Y must be nilpotent.

The Jordan form of $X = \begin{pmatrix} 0 & 0 & -\sqrt{-1} \\ 0 & 0 & 1 \\ \sqrt{-1} & -1 & 0 \end{pmatrix}$ can be calculated by Maple or ab-initio. We find that $e_1 = (1, 0, 0)^t$ is a cyclic generator and $Xe_1 = (0, 0, \sqrt{-1})^t$, $X^2e_1 =$

$(1, \sqrt{-1}, 0)^t$. Hence

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & \sqrt{-1} \\ 0 & \sqrt{-1} & 0 \end{pmatrix},$$

is the change of basis matrix. Therefore, under the change of variables $(\tilde{x}, \tilde{y}, \tilde{z})^t = C^{-1}(x, y, z)^t$, the vector field corresponding to X , namely $(\partial_x, \partial_y, \partial_z) X(x, y, z)^t$, transforms to

$$(\partial_{\tilde{x}}, \partial_{\tilde{y}}, \partial_{\tilde{z}}) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} (\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{X}.$$

This means that $\tilde{X} = \tilde{x} \partial_{\tilde{y}} + \tilde{y} \partial_{\tilde{z}}$. Its basic invariants, given by the method of characteristics, are \tilde{x} , $\tilde{y}^2 - 2\tilde{x}\tilde{z}$. In the space of homogeneous polynomials of degree $d \geq 2$, the null space \tilde{X} — for even values of d — is $\langle \tilde{x}^d, \tilde{x}^{d-2}(\tilde{y}^2 - 2\tilde{x}\tilde{z}), \tilde{x}^{d-4}(\tilde{y}^2 - 2\tilde{x}\tilde{z})^2, \dots, (\tilde{y}^2 - 2\tilde{x}\tilde{z})^{d/2} \rangle$, and the null space of \tilde{X} — for odd values of d — is

$$\langle \tilde{x}^d, \tilde{x}^{d-2}(\tilde{y}^2 - 2\tilde{x}\tilde{z}), \tilde{x}^{d-4}(\tilde{y}^2 - 2\tilde{x}\tilde{z})^2, \dots, \tilde{x}(\tilde{y}^2 - 2\tilde{x}\tilde{z})^{(d-1)/2} \rangle.$$

Moreover, as $(\partial_x, \partial_y, \partial_z)^t = (C^{-1})^t (\partial_{\tilde{x}}, \partial_{\tilde{y}}, \partial_{\tilde{z}})^t$, we have

$$\partial_x = \partial_{\tilde{x}}, \quad \partial_y = \sqrt{-1}(\partial_{\tilde{x}} - \partial_{\tilde{z}}), \quad \partial_z = \sqrt{-1} \partial_{\tilde{y}}.$$

As $x = \tilde{x} + \tilde{z}$, $y = \sqrt{-1} \tilde{z}$, $z = \sqrt{-1} \tilde{y}$, the vector field $I = y \partial_x - x \partial_y$ transforms to $\tilde{I} = \sqrt{-1}(-\tilde{x} \partial_{\tilde{x}} + (\tilde{x} + \tilde{z}) \partial_{\tilde{z}})$.

Now \tilde{I} operates on the null space of \tilde{X} , and

$$\tilde{I}(\tilde{x}) = -\sqrt{-1} \tilde{x}, \quad \tilde{I}(\tilde{y}^2 - 2\tilde{x}\tilde{z}) = -2\sqrt{-1} \tilde{x}^2.$$

Let $I_1 = -\sqrt{-1} \tilde{I}$. Thus

$$I_1(\tilde{x}) = -\tilde{x}, \quad I_1(\tilde{y}^2 - 2\tilde{x}\tilde{z}) = -2\tilde{x}^2.$$

Therefore, we have $I_1(\tilde{x}^a(\tilde{y}^2 - 2\tilde{x}\tilde{z})^b) = -a\tilde{x}^a(\tilde{y}^2 - 2\tilde{x}\tilde{z})^b - 2b\tilde{x}^{a+2}(\tilde{y}^2 - 2\tilde{x}\tilde{z})^{b-1}$.

If we set $e_{ab} = \tilde{x}^a(\tilde{y}^2 - 2\tilde{x}\tilde{z})^b$, then $I_1(e_{ab}) = -a e_{ab} - 2b e_{a+2b-1}$.

As a basis of the null space of \tilde{X} is $e_{d,0}, e_{d-2,2}, e_{d-4,4}, \dots$, the matrix of \tilde{X} is bidiagonal with eigenvalues $-d, -(d-2), -(d-4), \dots$ terminating in 0 or 1, depending on whether d is even or odd.

Notice that now our commutation relations read $[I_1, X] = X$, $[I_1, Y] = -Y$. Keeping in mind that the map from constant matrices to linear vector fields is an anti isomorphism, if $I_1 f = \lambda f$, then $I_1(Yf) = (\lambda + 1)f$. Thus each high weight vector with eigenvalue $-a$ (note that $a \geq 0$), gives a subspace of dimension $2a + 1$.

Moreover, the high weight vectors can be explicitly worked out, as the matrix of I_1 is bidiagonal with distinct eigenvalues. Some specific examples of high weight vectors in low dimensions and the corresponding real decompositions are given below.

5.2 (a) *Quadratic Polynomials:* The decomposition of homogeneous quadratic polynomials with complex coefficients in the variables x, y and z is

$$\langle x^2 + y^2 + z^2 \rangle \bigoplus \langle \mathrm{SO}(3, \mathbb{C})(x + \sqrt{-1}y)^2 \rangle.$$

These irreducible subspaces are obviously self conjugate — as they must be for the reasons given in Example 5.1. This gives the decomposition of real homogeneous polynomials of degree two in 3 variables as

$$\langle x^2 + y^2 + z^2 \rangle \bigoplus \langle \mathrm{SO}(3) \cdot (x^2 - y^2) \rangle = \langle x^2 + y^2 + z^2 \rangle \bigoplus \langle \mathrm{SO}(3) \cdot xy \rangle.$$

The equality comes by taking the real and imaginary parts, respectively of the highest weight vector.

5.2 (b) *Cubic Polynomials:* For homogeneous polynomials of degree 3, the null space of \tilde{X} is two dimensional and the eigen-space of \tilde{I} in this null space is given by the eigen polynomials \tilde{x}^3 and $\tilde{x}^3 + \tilde{x}\tilde{y}^2 - 2\tilde{x}^2\tilde{z}$: we need only the real or imaginary parts to get the decomposition of real homogeneous cubics under the action of $\mathrm{SO}(3)$. Working with the real parts of these weight vectors, we see that the decomposition of the space of real cubics under the action of $\mathrm{SO}(3)$ is

$$\langle \mathrm{SO}(3) \cdot (x^3 - 3xy^2) \rangle \bigoplus \langle \mathrm{SO}(3) \cdot x(x^2 + y^2 + z^2) \rangle.$$

5.2 (c) *Quartic Polynomials:* The eigen-space of \tilde{I} in the null space of \tilde{X} for homogeneous polynomials of degree 4 is given by the eigen polynomials \tilde{x}^4 , $-\tilde{x}^4 + \tilde{x}^2(\tilde{y}^2 - 2\tilde{x}\tilde{z})$ and $\tilde{x}^4 - 2\tilde{x}^2(\tilde{y}^2 - 2\tilde{x}\tilde{z}) + ((\tilde{y}^2 - 2\tilde{x}\tilde{z}))^2$. Working with the real parts of high weight vectors, the decomposition of the space of real quartics under the action of $\mathrm{SO}(3)$ is

$$\langle \mathrm{SO}(3) \cdot (x^4 - 6x^2y^2 + y^4) \rangle \bigoplus \langle \mathrm{SO}(3) \cdot ((y^2 - x^2)z^2 + y^4 - x^4) \rangle \bigoplus \langle \mathrm{SO}(3) \cdot (x^2 + y^2 + z^2)^2 \rangle.$$

The examples given below are similar because all the Lie algebras considered in these examples complexify to $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. However, we have given details in each case because we want to give an explicit description of the decomposition of real homogeneous polynomials of degree d under the action of $\mathfrak{so}(4)$, $\mathfrak{so}(1, 3)$ and $\mathfrak{so}(2, 2)$.

Example 5.3. Decomposition of real representations of $\mathfrak{so}(4)$ in the space of real homogeneous polynomials of degree d .

We will use a completely algebraic method which is suitable for calculations with software. It is based on Theorem 2.1 of [ABGM]. Using this result, one can determine the number of functionally independent invariants of any finite number of vector fields, without using the method of characteristics. If the vector fields have polynomial coefficients, one looks for functionally independent polynomial invariants, starting from degree 1 then degree 2 and so on till the requisite number of functionally independent invariants has been obtained. There is no guarantee that this process will always produce the requisite

number of invariants, but whenever it does, it is quite efficient. A basis of $\mathrm{SO}(4)$ is

$$\begin{aligned} e_1 &= e_{12} - e_{21}, e_2 = e_{13} - e_{31}, e_3 = e_{14} - e_{41}, \\ e_4 &= e_{23} - e_{32}, e_5 = e_{24} - e_{42}, e_6 = e_{34} - e_{43}, \end{aligned}$$

whose Cartan subalgebra is $C = \langle e_1, e_6 \rangle$. The roots of C are

$$a := (\sqrt{-1}, \sqrt{-1}), \quad b := (\sqrt{-1}, -\sqrt{-1}), \quad -a, \quad -b.$$

Thus the root system is of type $A_1 \times A_1$. Notice that conjugation maps every root to its negative.

The subalgebras generated by V_r, V_{-r} , $r = a, b$, are copies of $\mathfrak{sl}(2, \mathbb{C})$. As both these subalgebras are conjugation invariant, they must contain a copy of $\mathfrak{so}(3)$. Thus, the real and imaginary parts of the basis elements in V_a and V_b generate $\mathfrak{so}(4)$.

The real and imaginary parts of the root vectors in V_a and V_b are

$$I_1 = e_2 - e_5, \quad J_1 = e_3 + e_4, \quad I_2 = e_2 + e_5, \quad J_2 = e_3 - e_4.$$

Now $[I_1, J_1] = -2(e_1 + e_6) = -2K_1$ and $[I_2, J_2] = 2K_2$. The complexification of these two copies of $\mathfrak{so}(3)$ are generated by

$$\begin{aligned} X_1 &= \frac{J_1 - \sqrt{-1}K_1}{2}, \quad Y_1 = \frac{J_1 + \sqrt{-1}K_1}{2}, \\ X_2 &= \frac{J_2 - \sqrt{-1}K_2}{2}, \quad Y_2 = \frac{J_2 + \sqrt{-1}K_2}{2}, \end{aligned}$$

and they satisfy the commutation relations

$$\begin{aligned} [X_1, Y_1] &= \sqrt{-1}I_1, \quad [I_1, X_1] = -2\sqrt{-1}X_1, \quad [I_1, Y_1] = 2\sqrt{-1}Y_1, \\ [X_2, Y_2] &= \sqrt{-1}I_2, \quad [I_2, X_2] = -2\sqrt{-1}X_2, \quad [I_2, Y_2] = 2\sqrt{-1}Y_2. \end{aligned}$$

The generators of the Weyl group are

$$\omega_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, $\omega_1 I_1 \omega_1^{-1} = -I_1$, $\omega_2 I_2 \omega_2^{-1} = -I_2$, $\omega_2 I_1 \omega_2^{-1} = I_1$ and $\omega_1 I_2 \omega_1^{-1} = I_2$. The vector fields X_1 and X_2 are

$$\begin{aligned} X_1 &= \frac{(\sqrt{-1}y - w)}{2} \partial_x - \frac{(z + \sqrt{-1}x)}{2} \partial_y + \frac{(\sqrt{-1}w + y)}{2} \partial_z + \frac{(x - \sqrt{-1}z)}{2} \partial_w, \\ X_2 &= \frac{(y - \sqrt{-1}w)}{2} \partial_x - \frac{(z + \sqrt{-1}x)}{2} \partial_y - \frac{(\sqrt{-1}w + y)}{2} \partial_z + \frac{(x + \sqrt{-1}z)}{2} \partial_w, \end{aligned}$$

and they are of rank two. Thus, they must have two functionally independent invariants. Calculating the joint invariants in linear and then quadratic polynomials we find that these invariants are $e = w - \sqrt{-1}y$, $f = x^2 + z^2 + 2y(y + \sqrt{-1}w)$. As $de \wedge df$ is not identically zero, e and f are functionally independent joint invariants of X_1 and X_2 that generate all other invariants. Keeping in mind that e and f are of degrees 1 and 2 respectively, we obtain, in the space of homogeneous polynomials of degree $d \geq 2$, the subspace H of joint

invariants $\langle e^d, e^{d-2}f, e^{d-4}f^2, \dots, f^{d/2} \rangle$ for d even, and $\langle e^d, e^{d-2}f, e^{d-4}f^2, \dots, ef^{(d-1)/2} \rangle$ for d odd.

We have

$$I_1 = -z\partial_x + w\partial_y + x\partial_z - y\partial_w$$

and $I_1(e) = -\sqrt{-1}e$, $I_1(f) = 2\sqrt{-1}e^2$. Therefore,

$$I_1(e^a f^b) = -a\sqrt{-1}e^a f^b + 2b\sqrt{-1}e^{a+2}f^{b-1}.$$

Let $\sigma_{a,b} = e^a f^b$. The matrix of I_1 on H is bidiagonal with eigen-values

$$-d\sqrt{-1}, -(d-2)\sqrt{-1}, -(d-4)\sqrt{-1}, \dots \quad (5.1)$$

whereas the matrix of I_2 on H has eigen-values

$$d\sqrt{-1}, (d-2)\sqrt{-1}, (d-4)\sqrt{-1}, \dots \quad (5.2)$$

Thus if we have a high weight λ given by the values $(\lambda(I_1), \lambda(I_2))$, then $\lambda^\sigma = -\lambda$, and as

$$(\omega_1\omega_2)(I_1)(\omega_1\omega_2)^{-1} = -I_1, \quad (\omega_1\omega_2)(I_2)(\omega_1\omega_2)^{-1} = -I_2,$$

we see that $(\omega_1\omega_2)^{-1}\lambda^\sigma = \lambda$. Therefore, each of the irreducible components with high weights given by the sequences (5.1) and (5.2) are self-conjugate.

In the common null space of X_1 and X_2 in the space of homogeneous polynomials of degree d , the joint eigen-spaces of I_1 and I_2 give the decomposition of real homogeneous polynomials of degree d . These are given below explicitly for homogeneous polynomials of degree at most four: these were obtained by finding the eigen-vectors of the bidiagonal matrix of I_1 (or I_2) on H given above and taking their real parts.

a. Quadratic Polynomials:

$$\langle x^2 + y^2 + z^2 + w^2 \rangle \bigoplus \langle \text{SO}(4) \cdot (w^2 - y^2) \rangle.$$

b. Cubic Polynomials:

$$\langle \text{SO}(4) \cdot (w^3 - 3wy^2) \rangle \bigoplus \langle \text{SO}(4) \cdot w(x^2 + y^2 + z^2 + w^2) \rangle.$$

c. Quartic Polynomials:

$$\langle \text{SO}(4) \cdot (w^4 - 6w^2y^2 + y^4) \rangle \bigoplus \langle \text{SO}(4) \cdot (w^2 - y^2)(w^2 + x^2 + y^2 + z^2) \rangle \bigoplus \langle \text{SO}(4) \cdot (w^2 + x^2 + y^2 + z^2)^2 \rangle.$$

Example 5.4. Decomposition of representations of $\mathfrak{so}(1, 3)$ in real homogeneous polynomial of degree d .

A basis of $V = \mathfrak{so}(1, 3)$ is

$$e_1 = e_{12} + e_{21}, e_2 = e_{13} + e_{31}, e_3 = e_{14} + e_{41}, e_4 = e_{23} - e_{32}, e_5 = e_{24} - e_{42}, e_6 = e_{34} - e_{43}.$$

We have the Cartan subalgebra $C = \langle e_1, e_6 \rangle$. Note that there is no real split or compact Cartan subalgebra.

The roots of C are

$$a := (1, -\sqrt{-1}), b := (1, \sqrt{-1}), -a, -b.$$

The root system is of type $A_1 \times A_1$, with positive roots a, b , and conjugation maps a to b .

The real rank is one, and the eigenvalues of $\text{ad}(e_1)$ are $-1, -1, 0, 0, 1, 1$. The corresponding root spaces are

$$V_1 = \langle e_3 + e_5, e_2 + e_4 \rangle, V_{-1} = \langle -e_3 + e_5, -e_2 + e_4 \rangle.$$

Also, $[e_3 + e_5, -e_3 + e_5] = 2e_1$; the subalgebra generated by $e_3 + e_5, -e_3 + e_5$ is $\mathfrak{sl}(2, \mathbb{R})$, while the subalgebra generated by e_4, e_5, e_6 is $\mathfrak{so}(3)$. The positive and negative root vectors in the complexification of $\mathfrak{so}(1, 3)$ with respect to Cartan subalgebra C are X_1, Y_1 , and their conjugates X_2, Y_2 are given by

$$X_1 = e_3 + e_5 - \sqrt{-1}(e_2 + e_4), Y_1 = \frac{-e_3 + e_5 + \sqrt{-1}(-e_2 + e_4)}{4},$$

$$X_2 = e_3 + e_5 + \sqrt{-1}(e_2 + e_4), Y_2 = \frac{-e_3 + e_5 - \sqrt{-1}(-e_2 + e_4)}{4}.$$

Moreover, with $I_1 = -e_1 + \sqrt{-1}e_6$ and $I_2 = -e_1 - \sqrt{-1}e_6$ they satisfy the commutation relations

$$[X_1, Y_1] = I_1, [I_1, X_1] = -2X_1, [I_1, Y_1] = 2Y_1$$

and $[X_2, Y_2] = I_2, [I_2, X_2] = -2X_2, [I_2, Y_2] = 2Y_2$. This gives the decomposition of $\mathfrak{so}(1, 3)$ complexified as a direct sum of two copies of $\mathfrak{sl}(2, \mathbb{C})$ and its conjugate.

In terms of standard coordinates x, y, z and w , the vector fields X_1 and X_2 are given as

$$X_1 = (w - \sqrt{-1}z)\partial_x - (w - \sqrt{-1}z)\partial_y - \sqrt{-1}(x + y)\partial_z + (x + y)\partial_w,$$

$$X_2 = (w + \sqrt{-1}z)\partial_x - (w + \sqrt{-1}z)\partial_y + \sqrt{-1}(x + y)\partial_z + (x + y)\partial_w,$$

which are of rank two. Consequently, they have two functionally independent invariants. We find their joint invariants in linear and quadratic polynomials and these turn out to be $e = x + y$ and $f = w^2 + z^2 - 2x(x + y)$. Moreover, $de \wedge df \neq 0$, therefore e and f are functionally independent joint invariants of X_1 and X_2 that generate all other invariants. In the space of homogeneous polynomials of degree $d \geq 2$, we obtain the subspace H of joint invariants as $\langle e^d, e^{d-2}f, e^{d-4}f^2, \dots, f^{d/2} \rangle$ for d even, and $\langle e^d, e^{d-2}f, e^{d-4}f^2, \dots, f^{(d-1)/2} \rangle$ for d odd.

On the other hand the vector field I_1 is

$$I_1 = -y\partial_x - x\partial_y + \sqrt{-1}w\partial_z - \sqrt{-1}z\partial_w,$$

and $I_1(e) = -e$ and $I_1(f) = 2e^2$. Therefore, we have

$$I_1(e^a f^b) = -ae^a f^b + 2be^{a+2} f^{b-1}.$$

Let $\sigma_{a,b} = e^a f^b$. The matrix of I_1 on H is bidiagonal with eigen values

$$-d, -(d-2), -(d-4), \dots \quad (5.3)$$

Similarly, the matrix of I_2 on H has the same eigen values.

We now give decomposition of the space of real homogeneous polynomials under the action of $\mathrm{SO}(1, 3)$ for degree $d = 2, 3, 4$.

a. Quadratic Polynomials:

$$\langle \mathrm{SO}(1, 3) \cdot (x + y)^2 \rangle \bigoplus \langle \mathrm{SO}(1, 3) \cdot (x^2 - w^2 - y^2 - z^2) \rangle.$$

b. Cubic Polynomials:

$$\langle \mathrm{SO}(1, 3) \cdot (x + y)^3 \rangle \bigoplus \langle \mathrm{SO}(1, 3) \cdot (x + y)(x^2 - w^2 - y^2 - z^2) \rangle.$$

c. Quartic Polynomials:

$$\langle \mathrm{SO}(1, 3) \cdot (x + y)^4 \rangle \bigoplus \langle \mathrm{SO}(1, 3) \cdot (x^2 - w^2 - y^2 - z^2)^2 \rangle \bigoplus \langle \mathrm{SO}(1, 3) \cdot (x + y)^2 (x^2 - w^2 - y^2 - z^2) \rangle.$$

Example 5.5. Decomposition of real representations of $\mathfrak{so}(2, 2)$ in real homogeneous polynomial of degree d .

A basis of $V = \mathfrak{so}(2, 2)$ is

$$e_1 = e_{12} - e_{21}, e_2 = e_{13} + e_{31}, e_3 = e_{14} + e_{41}, e_4 = e_{23} + e_{32}, e_5 = e_{24} + e_{42}, e_6 = e_{34} - e_{43}.$$

A real split Cartan subalgebra is $C = \langle e_2, e_5 \rangle$, while a compact Cartan subalgebra is $\langle e_1, e_6 \rangle$.

The roots of C are

$$a := (1, 1), b := (1, -1), -a, -b.$$

The root spaces are

$$V_a = \langle e_1 - e_3 + e_4 - e_6 \rangle, V_b = \langle e_1 + e_3 + e_4 + e_6 \rangle$$

$$V_{-a} = \langle e_1 + e_3 - e_4 - e_6 \rangle, V_{-b} = \langle e_1 - e_3 - e_4 + e_6 \rangle.$$

Conjugation fixes the roots. Consequently, the subalgebra generated by the root spaces of a root and its negative is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Therefore, denoting the subalgebra generated by V_r, V_{-r} by $\langle V_r, V_{-r} \rangle$ the decomposition $\langle V_a, V_{-a} \rangle \oplus \langle V_b, V_{-b} \rangle$ gives an isomorphism of $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ with $\mathfrak{so}(2, 2)$.

The two copies of $\mathfrak{sl}(2, \mathbb{R})$ are

$$X_1 = e_1 - e_3 + e_4 - e_6, Y_1 = \frac{1}{4}(e_1 + e_3 - e_4 - e_6), I_1 = -e_2 - e_5,$$

$$X_1 = e_1 + e_3 + e_4 + e_6, Y_1 = \frac{1}{4}(e_1 - e_3 - e_4 + e_6), I_1 = -e_2 + e_5,$$

and they satisfy the relations

$$[X_1, Y_1] = I_1, [I_1, X_1] = -2X_1, [I_1, Y_1] = 2Y_1$$

and $[X_2, Y_2] = I_2, [I_2, X_2] = -2X_2, [I_2, Y_2] = 2Y_2$.

In terms of the standard coordinates x, y, z and w , the vector fields X_1 and X_2 are given as

$$\begin{aligned} X_1 &= -(y+w)\partial_x + (x+z)\partial_y + (y+w)\partial_z - (x+z)\partial_w, \\ X_2 &= -(y-w)\partial_x + (x+z)\partial_y + (y-w)\partial_z + (x+z)\partial_w, \end{aligned}$$

which are of rank two. Consequently, they have two functionally independent invariants. We find their joint invariants in linear and quadratic polynomials and these turn out to be $e = x+z$ and $f = w^2 - y^2 - 2x(x+z)$. Moreover, $de \wedge df \neq 0$, therefore e and f are functionally independent joint invariants of X_1 and X_2 that generate all other invariants. In the space of homogeneous polynomials of degree $d \geq 2$, we obtain the subspace H of joint invariants as $\langle e^d, e^{d-2}f, e^{d-4}f^2, \dots, f^{d/2} \rangle$ for d even, and $\langle e^d, e^{d-2}f, e^{d-4}f^2, \dots, f^{(d-1)/2} \rangle$ for d odd.

On the other hand the vector field I_1 is

$$I_1 = z\partial_x + w\partial_y + x\partial_z + y\partial_w,$$

and $I_1(e) = -e$ and $I_1(f) = 2e^2$. Therefore,

$$I_1(e^a f^b) = -ae^a f^b + 2be^{a+2} f^{b-1}.$$

Let $\sigma_{a,b} = e^a f^b$. The matrix of I_1 on H is bidiagonal with eigen values

$$-d, -(d-2), -(d-4), \dots \quad (5.4)$$

Similarly, the matrix of I_2 on H has the same eigen values.

Given below is the decomposition of homogeneous polynomials of degree $d = 2, 3, 4$ under the action of $\mathfrak{so}(2, 2)$.

a. Quadratic Polynomials:

$$\langle \mathrm{SO}(2, 2) \cdot (x+z)^2 \rangle \oplus \langle \mathrm{SO}(2, 2)(x^2 + y^2 - w^2 - z^2) \rangle.$$

b. Cubic Polynomials:

$$\langle \mathrm{SO}(2, 2) \cdot (x+z)^3 \rangle \oplus \langle \mathrm{SO}(2, 2) \cdot (x+z)(x^2 + y^2 - w^2 - z^2) \rangle.$$

c. Quartic Polynomials:

$$\langle \mathrm{SO}(2, 2) \cdot (x+z)^4 \rangle \oplus \langle \mathrm{SO}(2, 2) \cdot (x^2 + y^2 - w^2 - z^2)^2 \rangle \oplus \langle \mathrm{SO}(2, 2) \cdot (x+z)^2(x^2 + y^2 - w^2 - z^2) \rangle.$$

Example 5.6. The complexification of the adjoint representations $\mathfrak{so}(1, 3)$ and $\mathfrak{so}(4)$: These representations are already discussed in Examples 5.4 and 5.3 and they illustrate Lemma 4.6 and Lemma 4.8 respectively.

As shown in example 5.4, $\mathfrak{so}(1, 3)$ complexifies to two copies of $\mathfrak{sl}(2, \mathbb{C})$ — each one a conjugate of the other. The subalgebra $\langle X_1, X_2 \rangle$ is self centralizing. The highest weight vectors are thus X_1 and X_2 . Moreover, conjugation permutes the positive system of roots and the two high weights are conjugates of each other. The fixed points of conjugation are clearly $\mathfrak{so}(1, 3)$ and by Lemma 4.6, the fixed points of the conjugation map, namely

$\mathfrak{so}(1, 3)$, must be irreducible. On the other hand, for $\mathfrak{so}(4)$, the positive roots are mapped to negative roots and — using the notation of Section 4, $\omega^{-1}\lambda^\sigma$ equals λ for all high weights. Moreover the map $\omega\bar{\omega}$ is the identity map. Thus $\mathfrak{so}(4)$ must decompose as a direct sum of two $\mathfrak{so}(4)$ -invariant subspaces — invariant under the adjoint action of $\mathfrak{so}(4)$ — in confirmation with the explicit decomposition of $\mathfrak{so}(4)$ into a direct sum of two copies of $\mathfrak{so}(3)$.

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