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Supplementary material on derivatives and integrals involving trigonometric functions

Lemma: (a) $\lim_{x\to 0}\sin x = 0$ (b) $\lim_{x\to 0}\cos x = 1$

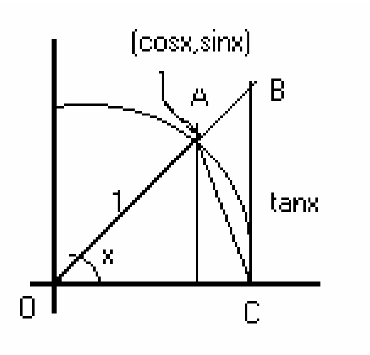
Theorem: $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Proof: In the unit circle of the graph, we have:
Area of triangle OAC <Area of the sector OAC <Area of triangle OBC

$$\frac{1}{2}\sin x < \frac{1}{2}r^2 x < \frac{1}{2}\tan x$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$1 > \frac{\sin x}{x} > \cos x$$



Using the sandwich theorem we conclude:

$$1 > \lim_{x\to 0}\frac{\sin x}{x} > \lim_{x\to 0}\cos x$$

That is $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Corollary: $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

Examples: Evaluate each of the following limits:

(1) $\lim_{x\to 0}\frac{\sin 3x}{x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x} = 3(1) = 3.$

(2) $\lim_{x\to 0}\frac{2x-\sin x}{x} = \lim_{x\to 0}(2-\frac{\sin x}{x}) = 2-1 = 1.$

(3) $\lim_{x\to 0}x\cot x = \lim_{x\to 0}\frac{x\cos x}{\sin x} = \lim_{x\to 0}\frac{\cos x}{(\sin x)/x} = 1$

Theorem:

(1) $\frac{d}{dx}(\sin x) = \cos x$.

(2) $\frac{d}{dx}(\cos x) = -\sin x$.

(3) $\frac{d}{dx}(\tan x) = \sec^2 x$.

(4) $\frac{d}{dx}(\cot x) = -\csc^2 x$

(5) $\frac{d}{dx}(\sec x) = \sec x\tan x$

(6) $\frac{d}{dx}(\csc x) = -\csc x\cot x$

<u>Theorem:</u>

(1) $\int \sin x\, dx = -\cos x + C$

(2) $\int \cos x\, dx = \sin x + C$

(3) $\int \sec^2 x\, dx = \tan x + C$

(4) $\int \csc^2 x\, dx = -\cot x + C$

(5) $\int \sec x \tan x\, dx = \sec x + C$

(6) $\int \csc x \cot x\, dx = -\csc x + C$

(7) $\int \tan x\, dx = \ln|\sec x| + C$

(8) $\int \cot x\, dx = \ln|\sin x| + C$

<u>The derivatives:</u> Let $u = g(x)$

(1) $\dfrac{d}{dx}(\sin u) = (\cos u)\dfrac{du}{dx}$.

(2) $\dfrac{d}{dx}(\cos u) = -(\sin u)\dfrac{du}{dx}$.

(4) $\dfrac{d}{dx}(\tan u) = (\sec^2 u)\dfrac{du}{dx}$.

(5) $\dfrac{d}{dx}(\cot u) = -(\csc^2 u)\dfrac{du}{dx}$

(6) $\dfrac{d}{dx}(\sec u) = (\sec u \tan u)\dfrac{du}{dx}$

(7) $\dfrac{d}{dx}(\csc u) = -(\csc u \cot u)\dfrac{du}{dx}$

<u>The integrals</u>

(1) $\int \sin u\, du = -\cos u + C$

(2) $\int \cos u\, du = \sin u + C$

(3) $\int \sec^2 u\, du = \tan u + C$

(4) $\int \csc^2 u\, du = -\cot u + C$

(5) $\int \sec u \tan u\, du = \sec u + C$

(6) $\int \csc u \cot u\, du = -\csc u + C$

(7) $\int \tan u\, du = \ln|\sec u| + C$

(8) $\int \cot u\, du = \ln|\sin u| + C$

<u>Problems</u>: Find the derivatives of the following.

1) $y = 3\cos x^2$

2) $y = 2x\sin^2 x$

3) $y = 2\sec x - x^2 \tan x$

4) $y = \dfrac{1-\cos x}{1+\sin x}$

5) $y = \cot x + x\csc^2 x$

6) $y = \ln(\cos x^2)$

7) $y = e^{\cos q}$

8) $y = \tan(e^x)$

9) $y = \ln(\sec x + \tan x)$

10) $y = \sqrt{\cos x}$

11) $y = \sin(\cos x)$

12) $y = 1 + \cot^2(2x)$

13) $y = \dfrac{1-\cot t}{\csc t}$

<u>Problems:</u> Evaluate the following integrals.

1) $\int \sin 2x\, dx$

2) $\int \sqrt{\sin q}\, \cos q\, dq$

3) $\int \dfrac{\sin x\, dx}{(1-\cos x)^4}$

4) $\int \dfrac{dx}{\cos^2 3x}$

5) $\int \dfrac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

6) $\int q \sin q^2\, dq$

7) $\int x \sin x\, dx$

8) $\int \dfrac{\sin(\ln x)}{x}\, dx$

9) $\int x \sec x^2 \tan x^2\, dx$