
Supplementary material on derivatives and integrals involving trigonometric functions

Lemma: (a) $\lim_{x \rightarrow 0} \sin x = 0$ (b) $\lim_{x \rightarrow 0} \cos x = 1$

Theorem: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Proof: In the unit circle of the graph, we have:

Area of triangle OAC < Area of the sector OAC < Area of triangle OBC

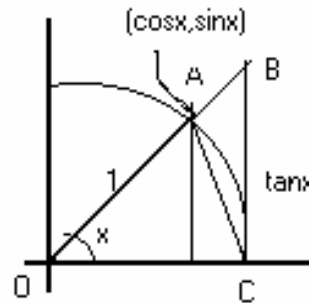
$$\frac{1}{2} \sin x < \frac{1}{2} r^2 x < \frac{1}{2} \tan x$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$1 > \frac{\sin x}{x} > \cos x$$

Using the sandwich theorem we conclude:

$$1 > \lim_{x \rightarrow 0} \frac{\sin x}{x} > \lim_{x \rightarrow 0} \cos x$$



That is $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

Corollary: $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$

Examples: Evaluate each of the following limits:

$$(1) \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = 3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} = 3(1) = 3.$$

$$(2) \lim_{x \rightarrow 0} \frac{2x - \sin x}{x} = \lim_{x \rightarrow 0} \left(2 - \frac{\sin x}{x} \right) = 2 - 1 = 1.$$

$$(3) \lim_{x \rightarrow 0} x \cot x = \lim_{x \rightarrow 0} \frac{x \cos x}{\sin x} = \lim_{x \rightarrow 0} \frac{\cos x}{(\sin x)/x} = 1$$

Theorem:

$$(1) \frac{d}{dx} (\sin x) = \cos x .$$

$$(3) \frac{d}{dx} (\tan x) = \sec^2 x .$$

$$(5) \frac{d}{dx} (\sec x) = \sec x \tan x$$

$$(2) \frac{d}{dx} (\cos x) = -\sin x .$$

$$(4) \frac{d}{dx} (\cot x) = -\csc^2 x$$

$$(6) \frac{d}{dx} (\csc x) = -\csc x \cot x$$

Theorem:

(1) $\int \sin x \, dx = -\cos x + C$

(3) $\int \sec^2 x \, dx = \tan x + C$

(5) $\int \sec x \tan x \, dx = \sec x + C$

(7) $\int \tan x \, dx = \ln |\sec x| + C$

(2) $\int \cos x \, dx = \sin x + C$

(4) $\int \csc^2 x \, dx = -\cot x + C$

(6) $\int \csc x \cot x \, dx = -\csc x + C$

(8) $\int \cot x \, dx = \ln |\sin x| + C$

The derivatives: Let $u = g(x)$

(1) $\frac{d}{dx}(\sin u) = (\cos u) \frac{du}{dx}$

(2) $\frac{d}{dx}(\cos u) = -(\sin u) \frac{du}{dx}$

(4) $\frac{d}{dx}(\tan u) = (\sec^2 u) \frac{du}{dx}$

(5) $\frac{d}{dx}(\cot u) = -(\csc^2 u) \frac{du}{dx}$

(6) $\frac{d}{dx}(\sec u) = (\sec u \tan u) \frac{du}{dx}$

(7) $\frac{d}{dx}(\csc u) = -(\csc u \cot u) \frac{du}{dx}$

The integrals

(1) $\int \sin u \, du = -\cos u + C$

(2) $\int \cos u \, du = \sin u + C$

(3) $\int \sec^2 u \, du = \tan u + C$

(4) $\int \csc^2 u \, du = -\cot u + C$

(5) $\int \sec u \tan u \, du = \sec u + C$

(6) $\int \csc u \cot u \, du = -\csc u + C$

(7) $\int \tan u \, du = \ln |\sec u| + C$

(8) $\int \cot u \, du = \ln |\sin u| + C$

Problems: Find the derivatives of the following.

1) $y = 3 \cos x^2$

2) $y = 2x \sin^2 x$

3) $y = 2 \sec x - x^2 \tan x$

4) $y = \frac{1 - \cos x}{1 + \sin x}$

5) $y = \cot x + x \csc^2 x$

6) $y = \ln(\cos x^2)$

7) $y = e^{\cos \theta}$

8) $y = \tan(e^x)$

9) $y = \ln(\sec x + \tan x)$

10) $y = \sqrt{\cos x}$

11) $y = \sin(\cos x)$

12) $y = 1 + \cot^2(2x)$

13) $y = \frac{1 - \cot t}{\csc t}$

Problems: Evaluate the following integrals.

1) $\int \sin 2x \, dx$

2) $\int \sqrt{\sin \theta} \cos \theta \, d\theta$

3) $\int \frac{\sin x \, dx}{(1 - \cos x)^4}$

4) $\int \frac{dx}{\cos^2 3x}$

5) $\int \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx$

6) $\int \theta \sin \theta^2 \, d\theta$

7) $\int e^{\sin x} \cos x \, dx$

8) $\int \frac{\sin(\ln x)}{x} \, dx$

9) $\int x \sec x^2 \tan x^2 \, dx$

10) $\int x \sin x \, dx$