KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICAL SCIENCES
MATH 260 -FINAL EXAM

TIME: 12:30 - 3:30 P.M. Tuesday – August 23, 2005

Serial Number:________________ Section Number:_______________

Name:_____________________________________ Student Number:_______________

Important Notes

1. Check that the exam paper has 10 different pages.
2. Do not use calculators.
3. Show all your work. No credits for answers not supported by work.

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1. (12 Points) The student population $U = U(t)$ at KFUPM has time rate of change $\frac{dU}{dt}$, which is proportional to the difference between maximum population of 20,000 and U. If there are 10,000 students this year and 11,000 next year, what will be the population in two years from now?

2. (12 Points) DO ONE PROBLEM (either a or b)

a. Find the solution of the initial value problem: $xy' = \frac{\cos x}{x} - 2y; \qquad y\left(\frac{\pi}{2}\right) = 0.$

b. Find the function $k(x)$ satisfying $k(0) = 0$ that makes the following equation an exact equation. $[3xy + xye^{x}]\,dx + [k(x) + \frac{3}{2}x^{2}]\,dy = 0.$

3. (12 Points) Use either the method of *undetermined coefficients OR variation of parameters* to find the general solution of the equation: $y'' - 2y' + y = 4xe^{x}$.

4. (8 Points) Suppose that $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are three linearly independent vectors in $\mathbb{R}^3$. Prove that the vectors $\{ \boldsymbol{x + y}, \boldsymbol{y + z}, \boldsymbol{x + z} \}$ are also linearly independent.

5. (8 Points) Consider the vectors $u = \begin{bmatrix} 1 \\ c \\ c \end{bmatrix}$ and $v = \begin{bmatrix} c \\ 1 \\ c \end{bmatrix}$. Find c so that $w = \begin{bmatrix} c \\ c \\ 1 \end{bmatrix}$ is a linear combination of u and v. Also write w as a linear combination of u and v.

6. (8 Points) Consider the system of equations $\begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & -3 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

   I. For what values of *a*, *b*, and *c* does the system has a unique solution?

   II. Use Cramer's Rule to find only the value of *x* when $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

7. (8 Points) Consider the equation $y''+3y'+2y=0$.

   I. Write the equation as a linear system of first order.

   II. Check that the two vectors $X_1(t)=\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ and $X_2(t)=\begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ generate the general solution of the system.

8. (12 Points) Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) Find the determinant of *A*.
(b) Find the eigenvalues and eigenvectors of *A*.
(c) Find an invertible matrix *P* and a diagonal matrix *D* such that $P^{-1}AP = D$. (DO NOT FIND P^{-1})
(d) Use Hamilton-Cayley theorem to find A^{-1}.

9. (12 Points) DO ONE PROBLEM (either a or b)

(a) Consider the linear system of differential equations $X' = AX$ where

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -4 & 3 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Given that 1 and $1+i$ are two eigenvalues of A with the corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1+i \\ 2i \end{bmatrix}$. Find the general solution of the system.

(b) Find the general solution of the system

$$x' = 4x + y$$
$$y' = -9x - 2y$$

Also find the solution to the initial value problem $x(0) = 1$ and $y(0) = 2$.

10. (14 Points) Find the general solution to the linear system $X' = AX$, where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 9 \\ 0 & -1 & -5 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$