KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematical Sciences

Methods of Applied Mathematics         MATH 301

Final Exam                            Summer Semester 2004/05

Time Allowed:   2Hours 30 minutes

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Please write clearly and show all work.

Instructor: F.D. Zaman
Q1) Verify Green’s theorem: \[ \oint_C P\,dx + Q\,dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \,dA \]

Where C is the boundary of region R formed by curves \( y = x \) and \( y = x^2 \). Also, \( P = x + y \), \( Q = 2x \). (5)
Q2) Solve the initial value problem using the Laplace transform

\[ y'' - 2y' - 1 = \delta(t - 2), \]

\[ y(0) = 0, \quad y'(0) = 1. \]
Q 3) Solve the partial differential equation using separation of variables
\[
\frac{\partial^2 u}{\partial x^2} - u = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0
\]
\[
u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0
\]
\[
u(x, 0) = f(x), \quad 0 < x < 1.
\]
Q 4) Solve the boundary value problem using the Fourier cosine transform

\[ \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}, \quad x > 0, t > 0 \]

\[ \frac{\partial u}{\partial x}(0, t) = 1, \quad t > 0 \]

\[ u(x, 0) = e^{-x}, \quad \frac{\partial u}{\partial t}(x, 0) = 0, x > 0. \]
Q 5) Solve the boundary value problem

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < R, \quad t > 0
\]

\[u(R,t) = 0, \quad t > 0\]

\[u(r,0) = f(r), \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0, \quad 0 < r < R\]

where \(a\) and \(R\) are positive constants.

Further, find the solution if \(f(r) = \begin{cases} 1, & 0 < r < R/2 \\ 0, & R/2 \leq r < R \end{cases}\)
Q6) Solve the Laplace equation in rectangle $0 < x < a$, $0 < y < b$ \[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b\]
\[\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(a, y) = 0, \quad 0 < y < b\]
\[u(x, 0) = f(x), \quad u(x, b) = 0.\]
Some formulae
1) The Bessel equation of order 0
\[ x^2 y'' + xy' - \lambda^2 x^2 y = 0 \]
has solution \( C_1 J_0(\lambda x) + C_2 Y_0(\lambda x) \)

2) Fourier Bessel Series: If the boundary condition is \( J_n(\lambda b) = 0 \)
\[
 f(x) = \sum_{i=1}^{n} A_i J_n(\lambda_i x)
\]
\[
 A_i = \frac{2}{b^2 J_{n+1}(\lambda_i b)} \int_0^b x J_n(\lambda_i x) f(x) dx
\]
\[
 \int r^n J_{n-1}(r) dr = r^n J_n(r) + C
\]

3) The Fourier cosine transform of \( f''(x) \)
\[
 F_c \{ f''(x) \} = -\alpha^2 F_c(\alpha) - f'(0)
\]