King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 535 [Functional Analysis I]
First Semester 2005–2006(051)

Exam I: October 26, 2005 **Time: 2 hours**

1. (a) Prove that the space $C[a, b]$ of all continuous functions from $I = [a, b]$ into reals is a Banach space under the norm

$$\|f\| = \max_{t \in I} |f(t)|.$$

   (b) Every metric on a vector space is not necessarily a norm. Justify this statement by means of a suitable example.

2. (a) Let E be a normed space and F be a Banach space. Prove that the vector space $L(E, F)$ of all linear and continuous transformations from E into F is a Banach space with respect to the norm

$$\|T\| = \sup_{\substack{x \in E \\ \|x\|=1}} \|Tx\|.$$

   (b) Define $f : C[a, b] \to (\mathbb{R}, \|\cdot\|)$ by

$$f(x) = \int_a^b x(t)dt.$$

   Calculate $\|f\|$.

3. (a) Define two equivalent norms on a vector space and show that any two norms on a finite-dimensional vector space are equivalent.

   (b) Explain how every finite dimensional subspace of a normed space is complete.

4. (a) Let p be a fixed integer such that $1 \leq p \leq \infty$. Define $\ell_p = \left\{ \{x = \{x_n\} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$. Consider ℓ_p under its usual sum norm and show that the dual of ℓ_p is ℓ_q where $\frac{1}{p} + \frac{1}{q} = 1$. Hence or otherwise find C_0^{**}.

   (b) Define a contraction on a metric space. Let $T : [1, \infty) \to [1, \infty)$ be given by $Tx = \frac{25}{26}\left(x + \frac{1}{x}\right)$. Show that T is a contraction.