1. (a) Prove that the space $C[a, b]$ of all continuous functions from $I = [a, b]$ into reals is a Banach space under the norm

$$\|f\| = \max_{t \in I} |f(t)|.$$ 

(b) Every metric on a vector space is not necessarily a norm. Justify this statement by means of a suitable example.

2. (a) Let $E$ be a normed space and $F$ be a Banach space. Prove that the vector space $L(E, F)$ of all linear and continuous transformations from $E$ into $F$ is a Banach space with respect to the norm $\|T\| = \sup_{x \in E} \|Tx\|.$

(b) Define $f : C[a, b] \to (\mathbb{R}, \|\cdot\|)$ by

$$f(x) = \int_a^b x(t)dt.$$ 

Calculate $\|f\|.$

3. (a) Define two equivalent norms on a vector space and show that any two norms on a finite-dimensional vector space are equivalent.

(b) Explain how every finite dimensional subspace of a normed space is complete.

4. (a) Let $p$ be a fixed integer such that $1 \leq p \leq \infty$. Define

$$\ell_p = \left\{ x = \{x_n\} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$ 

Consider $\ell_p$ under its usual sum norm and show that the dual of $\ell_p$ is $\ell_q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Hence or otherwise find $C_0^{**}.$

(b) Define a contraction on a metric space. Let $T : [1, \infty) \to [1, \infty)$ be given by $Tx = \frac{25}{26} \left( x + \frac{1}{x} \right).$ Show that $T$ is a contraction.