King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 535 [Functional Analysis I]
First Semester 2005–2006(051)

Exam II: December 24, 2005 **Time: 2 hours**

1. (a) Let X be a vector space over the field K and p a real-valued functional on X such that

$$p(x+y) \leq p(x) + p(y)$$

and

$$p(\alpha x) = |\alpha| p(x) \text{ where } x, y \in X \text{ and } \alpha \in K.$$

If f is a linear functional on a subspace Z of X satisfying $|f(x)| \leq p(x)$ for all $x \in Z$, then prove that f has an extension $\tilde{f}$ from Z to X such that $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

(b) For every x in a normed space X, show that

$$\|x\| = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|}.$$

2. (a) Let $\{T_n\}$ be a sequence of bounded linear transformations from a Banach space X into a normed space Y such that $\|T_n x\|$ is bounded for every $x \in X$. Then prove that the sequence $\{T_n\}$ is bounded.

(b) Let $\{\alpha_n\}$ be a sequence of reals. Define a sequence of functionals on ℓ_1 by

$$f_n(x) = \sum_{k=1}^{n} \alpha_k \xi_k, \quad x = \{\xi_k\} \in \ell_1.$$

Show that each f_n is linear and continuous and $\|f_n\| = \max_{1 \leq k \leq n} |\alpha_k|$. Assume that $\sum_{k=1}^{\infty} \alpha_k \xi_k$ is convergent for every $\{\xi_k\} \in \ell_1$. Use part (a), to show that $\{\alpha_n\}$ is bounded.

3. (a) Let T be a bounded linear mapping of a Banach space E into a normed space F. Suppose that there exists $\alpha > 0$ such that

$$\{y \in F : \|y\| \leq 1\} \subseteq \overline{T(B_\alpha)}$$

where $B_\alpha = \{x \in E : \|x\| \leq \alpha\}$. Show that there exists $\beta > 0$ such that

$$\{y \in F : \|y\| \leq 1\} \subseteq T(B_\beta).$$

(b) Let T be a bounded linear mapping of a Banach space E onto a Banach space F. If U is an open set in E, then prove by part (a) that $T(U)$ is an open subset of F.

4. (a) Prove that a closed linear mapping of a Banach space E into a Banach space F is continuous.

(b) Show by means of an example that if the completeness of E in part (a) is dropped, then T may not be continuous.