1. (a) Let $X$ be a vector space over the field $K$ and $p$ a real-valued functional on $X$ such that

$$p(x + y) \leq p(x) + p(y)$$

and

$$p(\alpha x) = |\alpha| p(x)$$

where $x, y \in X$ and $\alpha \in K$.

If $f$ is a linear functional on a subspace $Z$ of $X$ satisfying $|f(x)| \leq p(x)$ for all $x \in Z$, then prove that $f$ has an extension $\tilde{f}$ from $Z$ to $X$ such that $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

(b) For every $x$ in a normed space $X$, show that

$$\|x\| = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|}.$$ 

2. (a) Let $\{T_n\}$ be a sequence of bounded linear transformations from a Banach space $X$ into a normed space $Y$ such that $\|T_n x\|$ is bounded for every $x \in X$. Then prove that the sequence $\{T_n\}$ is bounded.

(b) Let $\{\alpha_n\}$ be a sequence of reals. Define a sequence of functionals on $\ell_1$ by

$$f_n(x) = \sum_{k=1}^{n} \alpha_k \xi_k, \quad x = \{\xi_k\} \in \ell_1.$$ 

Show that each $f_n$ is linear and continuous and $\|f_n\| = \max_{1 \leq k \leq n} |\alpha_k|$. Assume that $\sum_{k=1}^{\infty} \alpha_k \xi_k$ is convergent for every $\{\xi_k\} \in \ell_1$. Use part (a), to show that $\{\alpha_n\}$ is bounded.

3. (a) Let $T$ be a bounded linear mapping of a Banach space $E$ into a normed space $F$. Suppose that there exists $\alpha > 0$ such that

$$\{y \in F : \|y\| \leq 1\} \subseteq \overline{T(B_{\alpha})}$$

where $B_{\alpha} = \{x \in E : \|x\| \leq \alpha\}$. Show that there exists $\beta > 0$ such that

$$\{y \in F : \|y\| \leq 1\} \subseteq T(B_{\beta}).$$

(b) Let $T$ be a bounded linear mapping of a Banach space $E$ onto a Banach space $F$. If $U$ is an open set in $E$, then prove by part (a) that $T(U)$ is an open subset of $F$.

4. (a) Prove that a closed linear mapping of a Banach space $E$ into a Banach space $F$ is continuous.

(b) Show by means of an example that if the completeness of $E$ in part (a) is dropped, then $T$ may not be continuous.