King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 535 [Functional Analysis I]
First Semester 2005–2006(051)

Final Exam: January 22, 2006 **Time: 3 hours**

1. (a) Define a convex set in a normed space X. Show that the closed ball of X is convex.

   (b) Let p be a fixed integer such that $1 \leq p \leq \infty$. Define $\ell_p = \left\{ x = \{\alpha_n\} : \sum_{n=1}^{\infty} |\alpha_n|^p < \infty \right\}$.

   Prove that the space ℓ_p is a Banach space under the norm $\|x\| = \left(\sum_{n=1}^{\infty} |\alpha_n|^p \right)^{1/p}$.

2. (a) A closed and bounded subset of a finite dimensional normed space is compact. Verify this statement.

   (b) State and prove Hahn-Banach theorem for normed spaces.

3. (a) Let E be a Banach space, F a normed space and $\{T_n\}$ a sequence of continuous linear mappings from E into F. If $\lim_{n\to\infty} T_n x = Tx$ exists for each $x \in E$, then prove by the uniform boundedness principle that T is a continuous linear mapping of E into F.

   (b) Show that a one-to-one continuous linear mapping of a Banach space E onto a Banach space F is a linear homeomorphism of E onto F.

4. (a) Show that for x, y, z in an inner product space X, $\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\left\| z - \frac{x+y}{2} \right\|^2$.

   (b) Explain why the space $C[a, b]$ under its usual norm is not an inner product space.

5. (a) Let x^* be a bounded linear functional on a Hilbert space H. Prove that there exists a unique $y \in H$ such that $x^*(x) = \langle x, y \rangle$, for all $x \in H$ and $\|x^*\| = \|y\|$.

   (b) Use part (a), to show that every Hilbert space is reflexive.

6. (a) Verify that weak limit of a sequence in a normed space is unique. Also, give an example to show that a weakly convergent sequence may not be strongly convergent.

   (b) Define a complex Banach algebra A. Prove that the set of all invertible elements of A is an open subset of A. What conclusion can you draw from this about the set of all non-invertible elements of A?