1. (a) Define a convex set in a normed space $X$. Show that the closed ball of $X$ is convex.

(b) Let $p$ be a fixed integer such that $1 \leq p \leq \infty$. Define $\ell_p = \{x = \{\alpha_n\} : \sum_{n=1}^{\infty} |\alpha_n|^p < \infty\}$. Prove that the space $\ell_p$ is a Banach space under the norm $\|x\| = \left(\sum_{n=1}^{\infty} |\alpha_n|^p\right)^{1/p}$.

2. (a) A closed and bounded subset of a finite dimensional normed space is compact. Verify this statement.

(b) State and prove Hahn-Banach theorem for normed spaces.

3. (a) Let $E$ be a Banach space, $F$ a normed space and $\{T_n\}$ a sequence of continuous linear mappings from $E$ into $F$. If $\lim_{n \to \infty} T_n x = T x$ exists for each $x \in E$, then prove by the uniform boundedness principle that $T$ is a continuous linear mapping of $E$ into $F$.

(b) Show that a one-to-one continuous linear mapping of a Banach space $E$ onto a Banach space $F$ is a linear homeomorphism of $E$ onto $F$.

4. (a) Show that for $x, y, z$ in an inner product space $X$, $\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\left\|z - \frac{x + y}{2}\right\|^2$.

(b) Explain why the space $C[a, b]$ under its usual norm is not an inner product space.

5. (a) Let $x^*$ be a bounded linear functional on a Hilbert space $H$. Prove that there exists a unique $y \in H$ such that $x^*(x) = \langle x, y \rangle$, for all $x \in H$ and $\|x^*\| = \|y\|$.

(b) Use part (a), to show that every Hilbert space is reflexive.

6. (a) Verify that weak limit of a sequence in a normed space is unique. Also, give an example to show that a weakly convergent sequence may not be strongly convergent.

(b) Define a complex Banach algebra $A$. Prove that the set of all invertible elements of $A$ is an open subset of $A$. What conclusion can you draw from this about the set of all non-invertible elements of $A$?