Part 2: Essay Questions (1 hour)

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1. Tow cars start moving from the same point. One travels east at 40 km/h and the other travels west at 30 km/h. How fast is the distance between the cars increasing when they are 50 km away from each other?

\[ x_A \text{ represents the position of the car A} \]
\[ x_B \text{ represents the position of the car B} \]
\[ z = x_A - x_B \text{ (represents the distance between car A & B)} \]

\[ \frac{dx_A}{dt} = 40 \text{ km/h} \quad \text{&} \quad \frac{dx_B}{dt} = -30 \text{ km/h} \]

\[ \frac{dz}{dt} = \frac{dx_A}{dt} - \frac{dx_B}{dt} = 40 - (-30) = 70 \text{ km/h} \]

2. Let \( f(x) = x \tan x \) defined on the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\)

(a) Find the critical numbers of \( f \).
(b) Find the intervals on which \( f \) increases and decreases.
(c) Find the absolute maximum and minimum.

\[ f'(x) = \tan x + x \sec^2 x = \tan x + x(1 + \tan^2 x) = \frac{\sin x + \frac{x}{\cos x}}{\cos x} \]

\[ f'(x) = \frac{\sin x + \frac{x}{\cos^2 x}}{\cos x} \]

\[ f'(x) = 0 \implies x = 0 \text{ Critical points:} \]

\[ f'(x) > 0 \text{ if } x \in (0, \frac{\pi}{2}), \quad f'(x) < 0 \text{ if } x \in (-\frac{\pi}{2}, 0). \]

\[ \begin{array}{c|c|c}
-\frac{\pi}{2} & 0 & \frac{\pi}{2} \\
\hline
f & + \rightarrow & 0 & + \rightarrow \\
\hline
f' & + \rightarrow & 0 & + \rightarrow \\
\end{array} \]

- No absolute maximum.
- \( 0 \) absolute minimum.
3. Show that the equation \(4x^5 + x^3 + 2x + 1 = 0\) has exactly one real root.

The left hand side of the equation is a continuous function on \((\infty, +\infty)\). Since \(\lim_{x \to -\infty} f(x) = -\infty\) and \(\lim_{x \to +\infty} f(x) = +\infty\), we have at least one real root. (One can consider also the interval \([-1, 0]\) and notice that \(f(-1) = -6\) and \(f(0) = 1 > 0\) ...)

Let us call this root \(x_1\) and suppose that there exists another one \(x_2 \neq x_1\). As \(f(x_1) = f(x_2) = 0\), there exists (by Rolle's Thm) at least one \(c\) between \(x_1\) and \(x_2\) such that \(f'(c) = 0\). But, \(f'(x) = 20x^4 + 3x^2 + 2 > 2 > 0\) can never vanish. This is a contradiction and hence such a point \(x_2\) can not exist.

4. Use differentials (or linear approximation) to estimate \(\cot(134^\circ)\).

**Method (1) Using Linear Approx.**

Let \(f(x) = \cot x\), \(a = 135^\circ = \frac{3\pi}{4}\)

\[f(x) \approx f(a) + f'(a)(x-a)\]

\[f(x) = -\csc^2 x\]

\[\Rightarrow f'(\frac{3\pi}{4}) = -\left(\frac{2}{\sqrt{2}}\right)^2 = -2\]

\[\therefore f\left(\frac{3\pi}{4}\right) = -1\]

\[f(134^\circ) = f\left(\frac{3\pi}{4} - \frac{\pi}{180}\right)\]

\[\approx -1 - 2\left(\frac{3\pi}{4} - \frac{\pi}{180} - \frac{3\pi}{4}\right)\]

\[= -1 + \frac{3\pi}{180} \approx -1 + \frac{\pi}{90}\]

**Method (2) Using differentials**

\[f(a + dx) \approx f(a) + dy\] near \(a\)

Let \(f(x) = \cot x\), \(a = \frac{3\pi}{4}\), \(dx = -\frac{\pi}{180}\)

\[dy = f'(x) dx = -\csc^2 x dx\]

When \(a = \frac{3\pi}{4}\), \(dx = -\frac{\pi}{180}\)

\[dy = \left[-\csc^2\left(\frac{3\pi}{4}\right)\right] \left(-\frac{\pi}{180}\right)\]

\[= \frac{2\pi}{180} = \frac{\pi}{90}\]

\[f(134^\circ) = f\left(\frac{3\pi}{4} - \frac{\pi}{180}\right) = -1 + \frac{\pi}{90}\]
5. Let \( f(x) = xe^{-x} \).

Showing all details on the next empty page, find each of the following:

1. (a) Domain \( f(x) = \mathbb{R} = (-\infty, \infty) \)
   (b) \( x \)-intercept(s) (if any): 0
   (c) \( y \)-intercept (if any): 0
   (d) Symmetries (if any): None
   (e) \( \lim_{x \to +\infty} f(x) = 0 \)
   (f) \( \lim_{x \to -\infty} f(x) = -\infty \)
   (g) Asymptote(s) (if any): \( y = 0 \) is horizontal asymptote; No vertical asymptote
   (h) Critical Point(s) (if any): \( (1, \frac{1}{e}) \)
   (i) Interval(s) on which \( f(x) \) is increasing (if any): \( (-\infty, 1) \)
   (j) Interval(s) on which \( f(x) \) is decreasing (if any): \( (1, \infty) \)
   (k) Relative Maxima (if any): \( \frac{1}{e} \) at \( x = 1 \)
   (l) Relative Minima (if any):
   (m) Absolute Maximum (if any): \( \frac{1}{e} \) at \( x = 1 \)
   (n) Absolute Minimum (if any): None
   (o) Interval(s) on which the curve of \( y = f(x) \) is concave up (if any): \( (2, \infty) \)
   (p) Interval(s) on which the curve of \( y = f(x) \) is concave down (if any): \( (-\infty, 2) \)
   (q) Inflexion Point(s) (if any): \( (2, \frac{2}{e^2}) \)

2. Draw the graph of \( f(x) \) using the input found above: