Chapter
Derivatives and Integrals of Trigonometric Functions

In this Chapter we shall look at the derivatives and integrals of trigonometric functions. The derivative of the sine and cosine functions depend on the following limit theorems.

Lemma: (a) $\lim_{x\to 0}\sin x = 0$ (b) $\lim_{x\to 0}\cos x = 1$

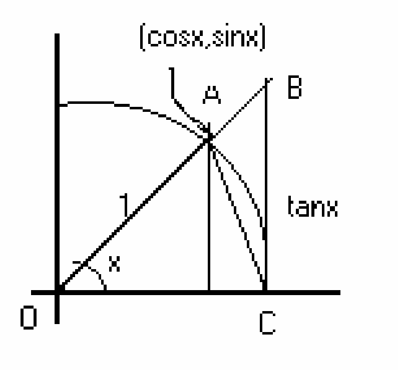
Theorem: $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Proof: In the unit circle of the graph, we have:
Area of triangle OAC <Area of the sector OAC <Area of triangle OBC

$$\frac{1}{2}\sin x < \frac{1}{2}r^2 x < \frac{1}{2}\tan x$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$1 > \frac{\sin x}{x} > \cos x$$



Using the sandwich theorem we conclude:

$$1 > \lim_{x\to 0}\frac{\sin x}{x} > \lim_{x\to 0}\cos x$$

That is $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Corollary: $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

Examples: Evaluate each of the following limits:

(1) $\lim_{x\to 0}\frac{\sin 3x}{x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x} = 3(1) = 3.$

(2) $\lim_{x\to 0}\frac{2x-\sin x}{x} = \lim_{x\to 0}(2-\frac{\sin x}{x}) = 2-1=1.$

(3) $\lim_{x\to 0} x\cot x = \lim_{x\to 0}\frac{x\cos x}{\sin x} = \lim_{x\to 0}\frac{\cos x}{(\sin x)/x} = 1$

Theorem:

(1) $\frac{d}{dx}(\sin x) = \cos x$.

(2) $\frac{d}{dx}(\cos x) = -\sin x$.

(3) $\frac{d}{dx}(\tan x) = \sec^2 x$.

(4) $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

(5) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

(6) $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

<u>Proof of Theorem:</u>

(1) $\dfrac{d}{dx}(\sin x) = \cos x$.

Proof:

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} =$$

$$= \lim_{\Delta x\to 0}\frac{\sin x\,\cos\Delta x + \cos x\,\sin\Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\sin x\,(\cos\Delta x - 1) + \cos x\,\sin\Delta x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\left[\frac{\sin x\,(\cos\Delta x - 1)}{\Delta x} + \frac{\cos x\,\sin\Delta x}{\Delta x}\right]$$

$$= 0 + \cos x = \cos x$$

<u>Theorem:</u>

(1) $\int \sin x\,dx = -\cos x + C$

(2) $\int \cos x\,dx = \sin x + C$

(3) $\int \sec^2 x\,dx = \tan x + C$

(4) $\int \csc^2 x\,dx = -\cot x + C$

(5) $\int \sec x \tan x\,dx = \sec x + C$

(6) $\int \csc x \cot x\,dx = -\csc x + C$

(7) $\int \tan x\,dx = \ln|\sec x| + C$

(8) $\int \cot x\,dx = \ln|\sin x| + C$

<u>The derivatives:</u> Let $u = g(x)$

(1) $\dfrac{d}{dx}(\sin u) = (\cos u)\,\dfrac{du}{dx}$.

(2) $\dfrac{d}{dx}(\cos u) = -(\sin u)\,\dfrac{du}{dx}$.

(3) $\dfrac{d}{dx}(\tan u) = (\sec^2 u)\,\dfrac{du}{dx}$.

(4) $\dfrac{d}{dx}(\cot u) = -(\csc^2 u)\,\dfrac{du}{dx}$

(5) $\dfrac{d}{dx}(\sec u) = (\sec u \tan u)\,\dfrac{du}{dx}$

(6) $\dfrac{d}{dx}(\csc u) = -(\csc u \cot u)\,\dfrac{du}{dx}$

The integrals

(1) $\int \sin u \, du = -\cos u + C$

(2) $\int \cos u \, du = \sin u + C$

(3) $\int \sec^2 u \, du = \tan u + C$

(4) $\int \csc^2 u \, du = -\cot u + C$

(5) $\int \sec u \tan u \, du = \sec u + C$

(6) $\int \csc u \cot u \, du = -\csc u + C$

(7) $\int \tan u \, du = \ln|\sec u| + C$

(8) $\int \cot u \, du = \ln|\sin u| + C$

Problems: Find the derivatives of the following.

1) $y = 3\cos x^2$
2) $y = 2x\sin^2 x$
3) $y = 2\sec x - x^2 \tan x$
4) $y = \dfrac{1-\cos x}{1+\sin x}$
5) $y = \cot x + x\csc^2 x$
6) $y = \ln(\cos x^2)$
7) $y = e^{\cos\theta}$
8) $y = \tan(e^x)$
9) $y = \ln(\sec x + \tan x)$
10) $y = \sqrt{\cos x}$
11) $y = \sin(\cos x)$
12) $y = 1 + \cot^2(2x)$
13) $y = \dfrac{1-\cot t}{\csc t}$

Problems: Evaluate the following integrals.

1) $\int \sin 2x \, dx$

2) $\int \sqrt{\sin\theta} \, \cos\theta \, d\theta$

3) $\int \frac{\sin x \, dx}{(1-\cos x)^4}$

4) $\int \frac{dx}{\cos^2 3x}$

5) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} \, dx$

6) $\int \theta \sin\theta^2 \, d\theta$

7) $\int x \sin x \, dx$

8) $\int \frac{\sin(\ln x)}{x} \, dx$

9) $\int x \sec x^2 \tan x^2 \, dx$

Ex. Try this problem: $\int e^x \sin x \, dx$