King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Semester II, 2005-2006(052)
MATH 202
Final Exam   June 4, 2006
Time: 3 hours

Student Name: ______________________________

Student ID: ______________________________

Section: ______________________________

Note:

FOR ALL PROBLEMS, SHOW WORK. NO CREDIT FOR ANSWERS NOT SUPPORTED BY WORK.

1. (a) For the differential equation $y'' - \omega^2 y = 0$, where ω is a positive constant, explain why the equation has a power series solution $y = \sum_{n=0}^{\infty} a_n x^n$.

   (b) Determine a recurrence relation for the constants a_n $(n = 0, 1, 2, \ldots)$.

   (c) Find a fundamental set of solutions using the recurrence relations in part (b).

(15 points)

2. (a) Find the singular points of the equation

$$y'' + \frac{x+1}{(x^3+x)}y' + \frac{y}{(x^2-4x+5)} = 0.$$

(b) If $y = \sum_{n=0}^{\infty} a_n(x-2)^n$ is a solution, and R is its radius of convergence, determine, without solving the equation a number L so that $R \geq L$. Justify your answer.

(10 points)

3. (a) Define what it means for x_0 to be a regular singular point of a differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

(b) Verify that the indicial equation of $2xy'' - y' + 2y = 0$ is $2r^2 - 3r = 0$.

(c) Give a recurrence formula for a solution $y = \sum a_n x^{n+r}$ of the equation in part (b)

(d) Give the first 4 terms of the solutions for the larger root of the indicial equation for the equation $2xy'' - y' + 2y = 0$.

(30 points)

4. (a) Find eigenvalues of $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) Find the general solution of the differential equation $dX/dt = AX$ where A is as in part (a)

(20 points)

5. Consider the system of differential equations

$$\frac{dx}{dt} = 2y - x\ , \ \frac{dy}{dt} = -x + y.$$

(a) Write the system in the form

$$\frac{dX}{dt} = AX, \quad \text{where} \quad X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) Find eigenvalues λ_1, λ_2 of A (hint: the eigenvalues are complex).

(c) Find two real fundamental solutions of the given system.

(15 points)

6. (a) Let $y_1(x) = x^3, \quad y_2(x) = |x|^3$. Are the functions y_1, y_2 linearly dependent on the interval $(-1, 1)$? Justify your answer? (8 points)

(b) Are y_1, y_2 linearly dependent on the interval $(0, 1)$? Justify your answer (2 points)

7. (a) When is a differential equation

$$P(x,y)dx + Q(x,y)dy = 0$$

an exact equation?

(b) Find integers m and n so that $\mu(x,y) = x^m y^n$ is an integrating factor of the equation

$$(-xy\sin x + 2y\cos x)dx + 2x\cos x\, dy = 0.$$

(15 points)

8. Solve the following initial value problem on the interval $0 \le x \le 2$:

$$\begin{aligned} \frac{dy}{dx} - y &= f(x) \\ y(0) &= 0 \end{aligned}$$

where $f(x) = \begin{cases} 1, & 0 \le x < 1 \\ -2, & 1 \le x \le 2 \end{cases}$ (10 points)

9. (a) Show that the inhomogeneous Cauchy-Euler equation

$$4x^2y'' + 8xy' + y = x + \ln x \quad (x > 0)$$

can be changed to the equation

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = t + e^t$$

by a suitable substitution.

(b) Give the general solution of the equation in part (a).

10. (a) Find the fundamental solutions of the system

$$\begin{aligned} 2\frac{dy}{dt} &= 0 \\ -\frac{dx}{dt} + 3\frac{dy}{dt} &= 0 \end{aligned}$$

(b) Find the fundamental matrix $\Phi(t)$ and its inverse.

(c) If $X_p = \phi(t)u(t)$ is a solution of the inhomogeneous equation

$$\frac{dX}{dt} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} X + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

show that

$$\frac{du}{dt} = [\phi(t)]^{-1} F(t),$$

where $F(t) = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$.

(20 points)