The objective of the exam is to prove the existence of at least one solution of the following boundary value problem

\[ \begin{cases} (p(t)u'(t))' + q(t)u(t) = f(t, u(t)), & 0 < t < \pi \\ u(0) = 0, & u(\pi) = 0, \end{cases} \quad (P) \]

by a topological method.
Notations.

Let $I$ denote the real interval $[0, \pi]$ and, for $k = 0, 1, \ldots C^k(I) = C^k(I; \mathbb{R})$ denotes the space of real-valued functions that are $k$-times continuously differentiable on $I$. For $u \in C^k(I)$, we define its norm by $\|u\| = \max\{|u(t)|; t \in I\}$. Then, $(C^k(I), \|\cdot\|)$ is a Banach space. Let $C^k_0(I) := \{u \in C^k(I); u(0) = u(\pi) = 0\}$.

Assumptions.

(H1) $p \in C^1(I)$, $p(t) \geq p_0 > 0$, $q \in C(I)$ and $q(t) \leq p_0$ with strict inequality on a subset of $I$ with positive measure.

(H2) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

there exists $r > 0$ such that $\int_0^\pi f(u(t))f(t, u(t))dt > 0$ whenever $|u(t)| > r$.

1. Show that $\int_0^\pi [p(t)u'(t)^2 - q(t)u(t)^2]dt < 0$ for all $u \in C^1_0(I)$
2. Show that the linear problem $(p(t)u'(t))' + q(t)u(t) = h(t)$, $u(0) = 0$, $u(\pi) = 0$ has a unique solution given by $u(t) = \int_0^\pi G(t, s)h(s)ds$, where $G(t, s)$ is the Green’s function corresponding to the linear homogeneous problem.
3. Consider the one-parameter family of problems

\[
\begin{aligned}
(p(t)u'(t))' + q(t)u(t) &= \lambda f(t, u(t)), & 0 < t < \pi \\
u(0) &= 0, & u(\pi) = 0.
\end{aligned}
\]

where $\lambda \in [0, 1]$. Show that $(P, \lambda)$ is equivalent to the integral equation

\[u(t) = \lambda \int_0^\pi G(t, s)f(s, u(s))ds\]

4. Use condition (H2) to show that all solutions of $(P, \lambda)$ are a priori bounded, independently of $\lambda$.

First, show that $\|u\|_0 \leq r$, $\|u'\|_0 \leq r_1 := \frac{\pi M_f + r \|q\|_0}{p_0}$, with $M_f = \max\{|f(t, u)|; t \in I, \|u\|_0 \leq r\}$.

Then $\|u''\|_0 \leq r_2 := r_1 (1 + \frac{\|p'\|_0}{p_0})$. Finally, conclude there exists $R > 0$ such that $\|u\| \leq R$.

5. Define an open, bounded, convex subset $\Omega$ of $C^2_0(I)$ by

$\Omega := \{u \in C^2_0(I); \|u\| < R + 1\}$.

Consider the map

\[H : [0, 1] \times \Omega \to C^2_0(I)\]

defined by

\[H(\lambda, u)(t) = \lambda \int_0^\pi G(t, s)f(s, u(s))ds\]
Show that, for each $\lambda \in [0, 1]$ \( H(\lambda, \cdot) : \bar{\Omega} \to C^2_0(I) \) is compact, and
\[
u \neq H(\lambda, u) \quad \forall u \in \partial \Omega, \; \lambda \in [0, 1].
\]

6. Show that the Leray-Schauder degree, \( d_{LS}(I - H(\lambda, \cdot), \Omega, 0) \) is well defined.
7. Conclude that the compact map \( H(1, \cdot) \) has at least one fixed point in \( \bar{\Omega} \),
and this fixed point is a solution of problem \( (P) \).