Write clearly. Marks may be deducted for messy work.

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1. Find all values of $a$ for which the vectors $\begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ a \end{bmatrix}, \begin{bmatrix} a \\ 3 \\ a \end{bmatrix}$ of $\mathbb{R}^3$ are linearly dependent.

2. Show that if two vectors $u, v$ form a basis of a vector space $V$, then $\{u + 2v, 2u - v\}$ is also a basis of $V$. 
3. Let \( L : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) be the linear transformation given by \( L \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ 0 \\ 0 \\ b \end{pmatrix} \).

(i) Find the matrix representation of \( L \) with respect to the basis \( S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \) of \( \mathbb{R}^3 \) and the basis \( T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \) of \( \mathbb{R}^4 \).

(ii) Find \( \ker L \). Is \( L \) onto? What is the rank of \( L \)?
4. Prove that for any two vectors $u, v$ in an inner product space: 

\[ \|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2. \]

5. We define a product in $P_2$ by

\[ (a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2) = a_0 b_0 + 2a_1 b_1 + 3a_2 b_2. \]

(i) Show that this is an inner product and find its matrix with respect to the basis \{1, 1 - t, 1 - t^2\} of $P_2$.

(ii) Find the cosine of the angle between $1 - t^2$ and $1 - t$ with respect to this inner product.
6. Let $A$, $B$, and $C$ be $n \times n$ symmetric matrices. Prove that

(i) $A$ is congruent to itself.

(ii) if $A$ is congruent to $B$ then $B$ is congruent to $A$.

(iii) if $A$ is congruent to $B$ and $B$ is congruent to $C$, then $A$ is congruent to $C$.

7. Let $A, B$ be $n \times n$ orthogonal matrices. Prove that $AB$ is orthogonal and that $\det A = \pm 1$. 
8. Let $g$ be a quadratic form in 3 variables given by $g(x) = -5x_1^2 + x_2^2 - x_3^2 + 4x_1x_2 + 6x_1x_3$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Find the canonical form of $g$ and determine its rank and its signature.

9. Is the quadratic form $3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$ (in 3 variables) positive definite? Justify.
10. Is the matrix \[
\begin{bmatrix}
5 & 0 & 1 \\
1 & 1 & 0 \\
-7 & 1 & 0
\end{bmatrix}
\] diagonalizable? If it is, diagonalize it, otherwise explain why it cannot be diagonalized.

11. Let \( A = \begin{bmatrix}
2 & 0 & 1 \\
0 & 3 & 0 \\
1 & 0 & 2
\end{bmatrix} \). Diagonalize \( A \) using an orthogonal matrix \( P \) (to be determined).
12. Label each of the following statements as TRUE or FALSE.

(i) If 2 matrices $A$ and $B$ are row equivalent and if $A$ is nonsingular then $B$ is nonsingular.
(ii) If a real matrix is skew-symmetric then it cannot have real eigenvalues.
(iii) If all the eigenvalues of $5 \times 5$ matrix are equal, then the matrix cannot be diagonalizable.
(iv) The vector \[
\begin{bmatrix}
1 \\
2 \\
4 \\
3
\end{bmatrix}
\] is in the subspace $W$ of $\mathbb{R}^4$ consisting of all vectors of the form \[
\begin{bmatrix}
a \\
b \\
c - b \\
c - a
\end{bmatrix}.
\]
(v) If a linear transformation $L : P_1 \rightarrow P_1$ has a matrix representation \[
\begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix},
\] then $L$ is an isomorphism.