9.2 The Directional Derivative

1) gradients
2) Gradient at a point
3) Vector differential operator
REV: Vectors in 2D & 3D

\[ \mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j} \]

**length** =

\[ \|\mathbf{u}\| = \sqrt{2^2 + 3^2} = \sqrt{13} \]

**Unit vector** =

\[ \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j} \]

**Dot product** =

\[ \mathbf{u} \cdot \mathbf{v} = (2\mathbf{i} + 3\mathbf{j}) \cdot (4\mathbf{i} - 2\mathbf{j}) = 2 \times 4 - 3 \times 2 = 2 \]
Gradient

\[ \nabla f (x, y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j = \text{del f} = \text{grad f} \]

**Example 1** gradients

Compute \( \nabla f (x, y) \) for

\[ f (x, y) = 5y - x^3 y^2 \]
Example 2  If \( F(x, y, z) = xy^2 + 3x^2 - z^3 \)

find \( \nabla F(x, y, z) \) at \((2, -1, 4)\)
Vector Differential Operator

2D
\[ \nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \]

3D
\[ \nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \]
Directional Derivative.

**Example 3** Find the directional derivative of \( f(x, y) = 2x^2 y^3 + 6xy \) at \((1,1)\) in the direction of

(A) the unit vector \( \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j \)

(B) a unit vector in the direction of \(3i+4j\)

(C) a unit vector whose angle with the positive x-axis is \(\frac{\pi}{6}\)

(D) The unit vector \(0i+j\)
Directional Derivative.

Geometric representation (1D)
In this section we will introduce a type of derivative, called a directional derivative.

Suppose that we wish to find the directional derivative of \( f \) at \((x_0, y_0)\) in the direction of an arbitrary unit vector \( u = <a, b> \)

To do this we consider the surface \( S \) with equation \( z=f(x,y) \)
And
we let \( z_0=f(x_0,y_0) \) then the point \( P(x_0,y_0,z_0) \) lies on \( S \).

The vertical plane that passes through \( P \) in the direction of \( u \) intersects \( S \) in a curve \( C \).

The slope of the tangent line \( T \) to \( C \) at \( P \) is the directional derivative of \( f \) at \((x_0,y_0)\) in the direction of \( u \)
Example 5 Find the directional derivative of \( F(x, y, z) = xy^2 - 4x^2y + z^2 \) at \((1, -1, 2)\) in the direction of:

\[ a) \mathbf{u} = \frac{1}{\sqrt{2}} \mathbf{i} + 0 \mathbf{j} + \frac{1}{\sqrt{2}} \mathbf{k} \]

\[ b) \mathbf{u} = 6 \mathbf{i} + 2 \mathbf{j} + 3 \mathbf{k} \]
Directional Derivative.

**Example 4** Directional Derivative

Consider the plane that is perpendicular to the \( xy \)-plane and passes through the points \( P(2, 1) \) and \( Q(3, 2) \). What is the slope of the tangent line to the curve of intersection of this plane with the surface \( f(x, y) = 4x^2 + y^2 \) at \( (2, 1, 17) \) in the direction of \( Q \)?

\[ \text{ezsurf('4*x^2+y^2',[0,4])} \]
Example 4  Directional Derivative

Consider the plane that is perpendicular to the $xy$-plane and passes through the points $P(2, 1)$ and $Q(3, 2)$. What is the slope of the tangent line to the curve of intersection of this plane with the surface $f(x, y) = 4x^2 + y^2$ at $(2, 1, 17)$ in the direction of $Q$?
In Problems 1–4, compute the gradient for the given function.

1. \( f(x, y) = x^2 - x^3 y^2 + y^4 \)
2. \( f(x, y) = y - e^{-2x^2y} \)
3. \( F(x, y, z) = \frac{xy^2}{z^3} \)
4. \( F(x, y, z) = xy \cos yz \)
In Problems 5–8, find the gradient of the given function at the indicated point.

5. \( f(x, y) = x^2 - 4y^2; \ (2, 4) \)

6. \( f(x, y) = \sqrt{x^3y} - y^4; \ (3, 2) \)

7. \( F(x, y, z) = x^2z^2 \sin 4y; \ (-2, \pi/3, 1) \)

8. \( F(x, y, z) = \ln(x^2 + y^2 + z^2); \ (-4, 3, 5) \)
In Problems 9 and 10, use Definition 9.5 to find $D_uf(x, y)$ given that $u$ makes the indicated angle with the positive $x$-axis.

9. $f(x, y) = x^2 + y^2; \quad \theta = 30^\circ$

10. $f(x, y) = 3x - y^2; \quad \theta = 45^\circ$
In Problems 11–20, find the directional derivative of the given function at the given point in the indicated direction.

11. $f(x, y) = 5x^3y^6; \ (-1, 1), \ \theta = \pi/6$

12. $f(x, y) = 4x + xy^2 - 5y; \ (3, -1), \ \theta = \pi/4$

13. $f(x, y) = \tan^{-1} \frac{y}{x}; \ (2, -2), \ \mathbf{i} - 3\mathbf{j}$
In Problems 21 and 22, consider the plane through the points $P$ and $Q$ that is perpendicular to the $xy$-plane. Find the slope of the tangent at the indicated point to the curve of intersection of this plane and the graph of the given function in the direction of $Q$.

21. $f(x, y) = (x - y)^2; \quad P(4, 2), \quad Q(0, 1); \quad (4, 2, 4)$

22. $f(x, y) = x^3 - 5xy + y^2; \quad P(1, 1), \quad Q(-1, 6); \quad (1, 1, -3)$
In Problems 27–30, find a vector that gives the direction in which the given function decreases most rapidly at the indicated point. Find the minimum rate.

27. \( f(x, y) = \tan(x^2 + y^2); \ (\sqrt{\pi/6}, \sqrt{\pi/6}) \)

28. \( f(x, y) = x^3 - y^3; \ (2, -2) \)

29. \( F(x, y, z) = \sqrt{xyz}^{e^y}; \ (16, 0, 9) \)

30. \( F(x, y, z) = \ln \frac{xy}{z}; \left( \frac{1}{2}, \frac{1}{6}, \frac{1}{3} \right) \)
Maximum Value of the Directional Derivative  Let $f$ represent a function of either two or three variables. Since (5) and (9) express the directional derivative as a dot product, we see from Definition 7.3 that

$$D_u f = ||\nabla f|| ||u|| \cos \phi = ||\nabla f|| \cos \phi, \quad (||u|| = 1),$$

where $\phi$ is the angle between $\nabla f$ and $u$. Because $0 \leq \phi \leq \pi$, we have $-1 \leq \cos \phi \leq 1$ and, consequently, $-||\nabla f|| \leq D_u f \leq ||\nabla f||$. In other words:
The maximum value of the directional derivative is $|\nabla f|$ and it occurs when $u$ has the same direction as $\nabla f$ (when $\cos \phi = 1$),

The minimum value of the directional derivative is $-|\nabla f|$ and it occurs when $u$ and $\nabla f$ have opposite directions (when $\cos \phi = -1$).
The gradient vector $\nabla f$ points in the direction in which $f$ increases most rapidly, whereas $-\nabla f$ points in the direction of the most rapid decrease of $f$. 
Example 7  Direction of Steepest Ascent

Each year in Los Angeles there is a bicycle race up to the top of a hill by a road known to be the steepest in the city. To understand why a bicyclist with a modicum of sanity will zigzag up the road, let us suppose the graph of \( f(x, y) = 4 - \frac{2}{3} \sqrt{x^2 + y^2} \), \( 0 \leq z \leq 4 \), shown in Figure 9.28(a) is a mathematical model of the hill. The gradient of \( f \) is
\[ \nabla f(x, y) = \frac{2}{3} \left[ \frac{-x}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{-y}{\sqrt{x^2 + y^2}} \mathbf{j} \right] = \frac{2/3}{\sqrt{x^2 + y^2}} \mathbf{r}, \]

where \( \mathbf{r} = -x \mathbf{i} - y \mathbf{j} \) is a vector pointing to the center of the circular base.

Thus the steepest ascent up the hill is a straight road whose projection in the \( xy \)-plane is a radius of the circular base. Since \( D_u f = \text{comp}_u \nabla f \), a bicyclist will zigzag, or seek a direction \( u \) other than \( \nabla f \), in order to reduce this component.