King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences

Math 311
First Major Exam
Semester I, 2006-2007(062)
April 1, 2007
8:00 p.m. – 10:00 p.m.

Name: ____ Grade __________/50

ID #: ____________________________

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1. Show all your work. Less credit will be given for answer not supported by proper work.
2. Clearly indicate the theorem or result you use.
3. This exam consists of 11 pages.

1. (a) Define (i) ordered field (ii) Archimedean field (iii) complete field. (3 points)

   (b) Give an example of a field which is not ordered. Include reasons in support of your claim. (2 points)

2. (a) In an ordered field prove or disprove:

If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$. (2 points)

(b) Prove that between any two real numbers, there is an irrational number. (3 points)

3. (a) If $a, b \in \mathbb{R}$, then show that maximum $\{-a, -b\} = -$ minimum$\{a, b\}$.
(2 points)

(b) For $a \in \mathbb{R}$, $a > 1$ and $n \in \mathbb{N}$, prove by means of mathematical induction that $a^n > 1 + n(a-1)$.
(3 points)

4. Let f be a function from a domain $D \subseteq \mathbb{R}$ to $\mathbb{R}$. Define continuity of f at $a \in D$. Prove by using definition that $f(x) = x^2$ is continuous on $\mathbb{R}$. (5 points)

5. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and $a \in \mathbb{R}$. Then prove that the following statements are equivalent:

   (i) f is continuous at a.

   (ii) If (x_n) is a sequence in $\mathbb{R}$ such that $x_n \to a$, then $f(x_n) \to f(a)$.

(3 points)

(b) Use part (a) to check whether or not the following function is continuous at $x = 0$.

$$f(x) = \begin{cases} \frac{1}{x} \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(2 points)

6. Define an open subset and a closed subset of $\mathbb{R}$. Prove that $[b, \infty)$ is closed.

(5 points)

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove that f is continuous if and only if $f^{-1}(H)$ is closed for every closed set H in $\mathbb{R}$. (5 points)

8. Explain how (i) a convergent sequence of real numbers is bounded (ii) the limit of a convergent sequence of non-negative real numbers is itself non-negative.

(5 points)

9. Let S be a nonempty bounded below subset of real numbers. Then prove that $\inf(S) = -\sup(-S)$. (5 points)

10. Show how a contraction $f : \mathbb{R} \to \mathbb{R}$ has a fixed point.

    A continuous function may not satify Lipschitz condition. Give an example in support of this statement.

    (5 points)