1. Show all your work. Less credit will be given for answer not supported by proper work.

2. Clearly indicate the theorem or result you use.

3. This exam consists of 9 pages.
1. Let $S$ be a subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ be a function. Define derivative of $f$ at a suitable point $x$ of $S$.

Let $f(x) = \begin{cases} 
  x^2 & \text{x is rational} \\
  0 & \text{x is irrational} 
\end{cases}$

Use definition to check whether $f(x)$ is differentiable at $x = 0$. (5 points)
2. Suppose that $g$ and $u$ are functions on $\mathbb{R}$ and $f(x) = g[u(x)]$. Suppose that $u$ has a derivative at $x_0$ and $g$ has a derivative at $u(x_0)$. Then prove that $f'(x_0)$ exists and $f'(x_0) = g'[u(x_0)]u'(x_0)$. (5 points)
3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(t) = \begin{cases} 
  t^2 \sin \frac{1}{t} & \text{if } t \neq 0 \\
  0 & \text{if } t = 0 
\end{cases}.$$ 

Show that $f$ is differentiable on $\mathbb{R}$ but $f'(x)$ is not continuous at $x = 0$. (5 points)
4. Suppose that $f$ is continuous on $[a, b]$, differentiable on $(a, b)$ and $f(a) = f(b)$. Prove that there exists a point $x_0$ in $(a, b)$ such that $f'(x_0) = 0$. What is the algebraic and geometric interpretation of this result. (5 points)
5. Examine the validity of the hypothesis and the conclusion of the mean value theorem for:

\[ f(x) = \begin{cases} 
\frac{x^2}{8} & 0 < x \leq 2 \\
2 & x = 0
\end{cases} \]

(5 points)
6. Suppose that $f$ and $F$ are continuous on $I = (a, b)$ and $f'$ and $F'$ exist with $F' \neq 0$ on $I$. If \( \lim_{x \to a^+} f(x) = \lim_{x \to a^+} F(x) = 0 \) and \( \lim_{x \to a^+} \frac{f'(x)}{F'(x)} = L \), then use mean value theorem to show that \( \lim_{x \to a^+} \frac{f(x)}{F(x)} = L \). (5 points)
7. Use suitable L’Hopital rules to calculate \( \lim_{x \to 0} \left( \frac{\tan x}{x} \right)^{1/x^2} \). (5 points)
8. If a differentiable function $f$ has an inverse function $g$ and if $f'(g(c)) \neq 0$, then prove that $g'(c) = \frac{1}{f'(g(c))}$. (5 points)