King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences

Math 311
Third Major Exam
Semester II, 2006-2007(062)
May 29, 2007
8:30 p.m. – 10:00 p.m.

Name: ______________________________ Grade ________/40

ID #: ____________________

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1. Show all your work. Less credit will be given for answer not supported by proper work.
2. Clearly indicate the theorem or result you use.
3. This exam consists of 9 pages.

1. Let S be a subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ be a function. Define derivative of f at a suitable point x of S.

   Let $f(x) = \begin{cases} x^2 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$.

   Use definition to check whether $f(x)$ is differentiable at $x = 0$. (5 points)

2. Suppose that g and u are functions on $\mathbb{R}$ and $f(x) = g[u(x)]$. Suppose that u has a derivative at x_0 and g has a derivative at $u(x_0)$. Then prove that $f'(x_0)$ exists and $f'(x_0) = g'[u(x_0)]u'(x_0)$. (5 points)

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(t) = \begin{cases} t^2 \sin \dfrac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases} .$$

Show that f is differentiable on $\mathbb{R}$ but $f'(x)$ is not continuous at $x = 0$. (5 points)

4. Suppose that f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$. Prove that there exists a point x_0 in (a, b) such that $f'(x_0) = 0$. What is the algebraic and geometric interpretation of this result. (5 points)

5. Examine the validity of the hypothesis and the conclusion of the mean value theorem for:

$$f(x) = \begin{cases} \dfrac{x^2}{8} & 0 < x \leq 2 \\ 2 & x = 0 \end{cases} .$$

(5 points)

6. Suppose that f and F are continuous on $I = (a, b)$ and f' and F' exist with $F' \neq 0$ on I. If $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} F(x) = 0$ and $\lim_{x \to a^+} \frac{f'(x)}{F'(x)} = L$, then use mean value theorem to show that $\lim_{x \to a^+} \frac{f(x)}{F(x)} = L$. (5 points)

7. Use suitable L'Hopital rules to calculate $\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^2}$. (5 points)

8. If a differentiable function f has an inverse function g and if $f'(g(c)) \neq 0$, then prove that $g'(c) = \dfrac{1}{f'(g(c))}$. (5 points)