King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 311

Final Exam

Semester II, 2006-2007(062)

June 11, 2007

7:30 a.m. – 10:30 a.m.

Name: ____ Grade __________/40

ID #: ____________________________

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1. Show all your work. Less credit will be given for answer not supported by proper work.
2. Clearly indicate the theorem or result you use.
3. This exam consists of 13 pages.

1. Suppose that $\lim_{x \to a} f_1(x) = L_1$ and $\lim_{x \to a} f_2(x) = L_2$. Then prove that $\lim_{x \to a}[f_1(x) \cdot f_2(x)] = L_1 L_2$. (3 points)

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $A \subseteq \mathbb{R}$. Prove that $f(\overline{A}) \subseteq \overline{f(A)}$ where — denotes closure of a st. Consider $A = Q$, the set of rationals and the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & x \text{ is rational} \\ 1 & x \text{ is irrational} \end{cases}$$

to check whether or not $f(\overline{A}) \subseteq \overline{f(A)}$ holds. (4 points)

3. Give an example to show that an infinite union of closed sets in $\mathbb{R}$ need not be closed. (3 points)

4. Let A be a bounded and closed subset of $\mathbb{R}$. Prove that $\sup(A)$ and $\inf(A) \in A$.
(3 points)

5. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function and $f(a) < c < f(b)$ for some $c \in \mathbb{R}$. Then prove that there is at least one $x_0 \in [a, b]$ such that $f(x_0) = c$.

(3 points)

6. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. Prove that the range of f is bounded. (3 points)

7. Show that $f(x) = x^2$ is not uniformly continuous on $[1, \infty)$. (3 points)

8. Let f and F be continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) \neq 0$ for all $x \in (a, b)$. Then show that (6 points)

   (a) $F(b) - F(a) \neq 0$.

   (b) there exists $\xi \in (a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = \dfrac{f'(\xi)}{F'(\xi)}$

   (c) there exist x_1, x_2 with $a < x_1 < x_2 < b$ and $\dfrac{1}{f'(x_1)} + \dfrac{1}{f'(x_2)} = 2$

   provided $f(a) = a$ and $f(b) = b$.

9. Define Darboux integral of a bounded real-valued function f on $[a, b]$. Use a suitable result to show that if f is increasing, then it is integrable. (3 points)

10. If f and g are continuous on $[a, b]$ and g is differentiable with $g'(x) = f(x)$ for each $x \in (a, b)$, then prove that $\int_a^b f(x)dx = g(b) - g(a)$. (3 points)

11. If f is Riemann integrable on $[a, b]$, then prove that f is Darboux integrable.

(3 points)

12. Prove that (3 points)

(a) $\log(1+x) \leq x$ for $x > -1$.

(b) $\dfrac{1}{2} \leq \log 2 \leq 1$.