Exercise 1 (15 points)
Let $G$ be an abelian group and $n$ a positive integer. Set $G_n = \{ x \in G / x^n = e \}$ and $H = \{ x^n / x \in G \}$. Prove that $G / G_n$ is isomorphic to $H$. (Hint, use First Isomorphism Theorem for groups).

Exercise 2 (20 points)
Let $R$ be a commutative ring with characteristic 3 and $\phi : R \rightarrow R$ defined by $\phi(x) = x^3$.

1-Prove that $\phi$ is a ring homomorphism.

2-Prove that for each positive integer $n$, and for each $x, y$ in $R$, $x^{3^n} + y^{3^n} = (x + y)^{3^n}$.

3-Find an example of a ring $A$ with characteristic 4 and two elements $x, y$ such that $(x + y)^4 \neq x^4 + y^4$. 
**Exercise 3** (20 points)
Let $K$ be a field and $\phi : K \to K$ be a ring homomorphism.
Prove that either $\phi$ is one-to-one or $\phi$ is the null homomorphism.

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**Exercise 4** (20 points)
Let $R$ be a commutative ring with unity and $I$ and $J$ two ideals of $R$.
1-Prove that $I \cap J$ is an ideal of $R$.
2-Prove that $IJ \subseteq I \cap J$.
3-Suppose that $I + J = R$. Prove that $IJ = I \cap J$. 
Exercise 5 (20 points)
Let $R$ and $S$ be a commutative rings with unities, $A$ an ideal of $S$ and $\phi: R \rightarrow S$ be a ring homomorphism. Prove that:
1-If $A$ is a prime ideal of $S$, then $\phi^{-1}(A)$ is a prime ideal of $R$.
2-If $A$ is a maximal ideal of $S$, then $\phi^{-1}(A)$ is a maximal ideal of $R$. 