Exercise 1 (30 points)
1- Let $G$ be a group.
   1- Prove that if $(ab)^2 = a^2b^2$, then $ab = ba$.
   2- Suppose that $G$ satisfies the following property: For every elements $a$, $b$ and $c$ of $G$, $ab = ca \Rightarrow b = c$. Prove that $G$ is abelian.

2- Let $G$ be an abelian group and $n$ a positive integer. Set $G_n = \{x \in G / x^n = e\}$ and $H = \{x^n / x \in G\}$. Prove that $G/G_n$ is isomorphic to $H$. (Hint, use First Isomorphism Theorem for groups).
Exercise 2 (30 points)
Let $G$ be a group and $H$ the subgroup of $G$ generated by the set $S = \{x^{-1}y^{-1}xy, x, y \in G\}$.

1. Prove that $H$ is a normal subgroup of $G$.
2. Prove that $G/H$ is a abelian.
3. Let $N$ be a subgroup of $G$. Prove that if $G/N$ is abelian, then $H \leq N$.
4. Prove that if $N$ is a subgroup of $G$ and $H \leq N$, then $N$ is normal
Exercise 3 (25 points)
Let $K$ be a field and $\phi : K \to K$ be a ring homomorphism.
Prove that either $\phi$ is one-to-one or $\phi$ is the null homomorphism.
Exercise 4 (30 points)
Let $R$ be a ring and $P$ an ideal of $R$.

1- Prove that $P$ is prime if and only if $R/P$ is an integral domain.

2- Prove that $P$ is maximal if and only if $R/P$ is a field.

3- In the polynomial ring $\mathbb{Z}[X]$ which one of the following ideals is prime or maximal (justify) $I = (2,X), J = (X)$
Exercise 5 (25 points)
Show whether the following polynomials are reducible or irreducible.

1- $f(X) = 2 + 4X + 6X^3 - 9X^5$ over $Z$.

2- $g(X) = 3X^2 + 4X + 3$ over $Z_5$
Exercise 6 (30 points)

Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a Principal Ideal Domain.

(Hint: Over a PID, every irreducible is prime)
Exercise 7 (30 points)
Construct a domain $R$ other than $\mathbb{Z}[\sqrt{-3}]$ and an element $x$ such that $x$ is irreducible but not prime.