1. Write clearly.

2. Show all your steps.

3. No credit will be given to wrong steps.

4. Do not do messy work.

5. Calculators and mobile phones are NOT allowed in this exam.
1. (6 Marks) Consider the following DE

\[ xy' - 3y = x^3 \ln x, \quad x > 0. \]

(a) Find the integrating factor.

(b) Solve the DE (1) subject to the initial condition \( y(1) = 1. \)
2. (8 Marks) Consider the DE

\[ \cos x \, dx + \left( 1 + \frac{2}{y} \right) \sin x \, dy = 0. \]

(a) Find an appropriate integrating factor to convert (1) to exact DE.
(b) Solve the DE (1).
3. (8 Marks) Solve the following DE

\[ y''' + y'' + 9y' + 9y = \begin{cases} 0 & 0 < t < 1, \\ e^{-3(t-1)} & t \geq 1. \end{cases} \]
4. (5 Marks) Find the general solution of the Cauchy-Euler DE

\[ xy^{(4)} + 6y''' = 0. \]
5. (4 Marks) Without solving the DE, find a lower bound of the radius of convergence of power series solutions of the DE

\[(x^2 - 2x - 15)y'' + 2xy' + y = 0,\]

about the ordinary point \(x = 2.\)
6. (a) (6 Marks) Find the exponential matrix for 

\[ A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

(b) If \( B \) is a nilpotent matrix, prove that \( \det(B) = 0 \).
7. (7 Marks) Let \( A \) be a \( 3 \times 3 \) matrix defined by

\[
A = \begin{bmatrix}
3 & -2 & 2 \\
0 & 1 & 0 \\
-1 & 1 & 0 \\
\end{bmatrix}
\]

(a) Find the eigenvalues of \( A \).
(b) Find the corresponding eigenvectors of \( A \).
(c) Find the general solution for the homogeneous system of DEs: \( X'(t) = AX(t) \).
8. (4 Marks) Use **Matrix Exponential** to find the general solution of the nonhomogeneous linear system

\[
X' = \begin{bmatrix}
-1 & 0 & 0 \\
0 & \ln(2) & 0 \\
0 & 0 & \pi \\
\end{bmatrix} X + \begin{bmatrix}
0 \\
-1 \\
e^{-t} \\
\end{bmatrix}
\]
9. (10 Marks) Consider the homogeneous system

\[ \begin{align*}
\frac{dx}{dt} &= 2x + 4y \\
\frac{dy}{dt} &= -x + 6y
\end{align*} \]

(a) Write the homogeneous linear system in matrix form.

(b) Find the general solution of (1).

(c) Use the method of variation of parameters to find a particular solution \( X_p \) of the nonhomogeneous system.

\[ \begin{align*}
\frac{dx}{dt} &= 2x + 4y - 9e^t \\
\frac{dy}{dt} &= -x + 6y
\end{align*} \]
10. (12 Marks) Consider the following DE

\[ 4xy'' + 2y' + y = 0, \]

(a) Find the regular singular point(s).

(b) Let \( y = \sum_{n=0}^{\infty} c_n x^{n+r} \). Show that

\[ r(2r - 1) = 0, \quad \text{and} \quad c_{k+1} = \frac{-c_k}{2(k + r + 1)(2k + 2r + 1)}, \quad \text{for } k \geq 0, \]

(c) For \( r = \frac{1}{2} \), use the recurrence relation in the part (b) to find \( c_k \) for \( k \geq 0 \) in terms of \( c_0 \).

(d) Using the above information, find a power series solution of the given DE about 0.

(e) For \( r = 0 \), use the recurrence relation in the part (b) to find \( c_k \) for \( k \geq 0 \) in terms of \( c_0 \).