KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Major 2
(Math 201)

Time Allowed: 1 ¼ Hour

Student Name:_______________  Id. No.______________

Section:______________

Note

No programmable calculators and mobile phones allowed in the examination hall. For all questions show calculations in support of your answers.

<table>
<thead>
<tr>
<th>Question</th>
<th>Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/3</td>
</tr>
<tr>
<td>2</td>
<td>4/4</td>
</tr>
<tr>
<td>3</td>
<td>4/4</td>
</tr>
<tr>
<td>4</td>
<td>4/4</td>
</tr>
<tr>
<td>5</td>
<td>3/3</td>
</tr>
<tr>
<td>6</td>
<td>2/2</td>
</tr>
<tr>
<td>Total</td>
<td>/20</td>
</tr>
</tbody>
</table>

Ashfaque H. Bokhari
Q1. Near a buoy at KFUPM sea beach water is \( z = 10 - 0.5x + 0.05y^2 \) deep. A swimmer starts at the point \((10, 8)\) and moves towards the buoy located at \((0, 0)\). Use directional derivatives to tell if the depth of water is increasing or decreasing towards the point \((10, 8)\). 

Points (2, 2)

\[ z = f(x, y) = 10 - 0.5x + 0.05y^2 \]

\[ \nabla f = (-0.5, 0.1y) \]

\[ \nabla f \big|_{(10,10)} = (-0.5, 1) = (-0.5, 1) \]

\[ \overrightarrow{A} = (0 - 10, 0 - 10) = (-10, -10) \]

\[ \hat{u} = \frac{-1}{10 \sqrt{2}} (0, 10) = \frac{-1}{\sqrt{2}} (1, 1) \]

\[ \text{Directional Derivative} = D_u f = \nabla f \cdot \hat{u} = \frac{-1}{\sqrt{2}} (1, 1) \cdot (-0.5, 1) = \frac{1}{\sqrt{2}} (0.5 - 1) = \frac{-1}{2\sqrt{2}} \]

Since the answer is positive, the depth is increasing under the boat.
Q2. Find an equation of (a) the tangent plane and (b) the normal line to the surface \( x^2 + y^3 + z = 5 \) at the point \((1, 1, 2)\).

Points (1.5, 1.5)

(a):

✓ Equation of the level surface is: \( F = x^2 + y^3 + z = 5 \)

✓ Therefore: \((F_x, F_y, F_z) = (2x, 3y^2, 1)\)

✓ Gradient at \((1, 1, 2)\): \((F_x, F_y, F_z) = (2, 3, 1)\)

✓ Equation: \(2(x - 1) + 3(y - 1) + 1(z - 2) = 0\)

(b):

✓ Gradient at \((1, 1, 1)\): \((F_x, F_y, F_z) = (2, 3, 1)\)

✓ Equation of normal line: \(\frac{x - 1}{2} = \frac{y - 1}{3} = \frac{z - 1}{1}\)
Q3

(a): Given \( z = \arctan(x + y) \), with \( x = st \), \( y = s - t \), find \( \partial z / \partial t \). Points (1)

(b): Recall the formula: Given \( F(x, y, z) = 0 \to \frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \). Use it to compute \( \frac{\partial z}{\partial x} \) when \( x y z^2 = \sin(xyz) \). Points(1)

(c): Use another method to verify your answer in (b). Points (1)

(a):

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \frac{1}{\sqrt{1 + (x + 2y)}} s + \frac{2}{\sqrt{1 + (x + 2y)}} (-1)
\]

\[
= \frac{s - 2}{\sqrt{1 + (x + 2y)}}
\]

(b)

\[ F(x, y, z) = x y z^2 - \sin(xyz) \]
\[ F_x = y z^2 - \cos(xyz) y z \]
\[ F_z = 2x y z - \cos(xyz) x y \]
\[ \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y z^2 + \cos(xyz) y z}{2x y z - \cos(xyz) x y} \]

(c)

\[ x y z^2 = \sin(xyz) \to x z^2 + 2x y z \frac{\partial z}{\partial x} = \cos(xyz) \{ y z + x y \frac{\partial z}{\partial x} \} \]
\[ \text{Re-arrange terms to get:} \quad \frac{\partial z}{\partial x} = \frac{-y z^2 + \cos(xyz) y z}{2x y z - \cos(xyz) x y} \]
Q4. Find linear approximation of the function \( f(x, y) = x - 5y^2 \) at the point (1, 3). Use it to approximate \( f(1.1, 2.9) \).

Linear Approximation: \( L = f(a, b) + f_x(x - a) + f_y(y - b) \)

\[ f(a, b) = 1 - 45 = -44, \quad f_x \big|_{(1,3)} = 1, \quad f_y \big|_{(1,3)} = -30 \]

The liner approximation is:
\[ L = -44 + (x - 1) - 30(y - 3) = x - 30y + 45 \]

\( L \big|_{(1,3)} = 1 - 90 + 45 = 44 \) and \( f(1,3) = 1 - 45 = 44 \)

\[ f(1.1, 2.9) = 1 - 5(2.9)^2 = -41.05; \quad L(1.1, 2.9) = 1.1 - 30 \times 2.9 + 45 = -40.9 \]

Not bad approximation
Q5. Given function \( z = (\ln x) y \), does Clairaut’s rule satisfy everywhere? In not, where this rule fails?  

Points (1)

- \( z_{xy} = z_{yx} \) is Clairaut’s rule
- \( z_x = \frac{y}{x} \)
- \( z_{xy} = \frac{1}{x} \)
- The derivative does not exist at \( x = 0 \), thus Clairaut’s rule is satisfied everywhere except where \( x \) is zero.

Q6. Volume of a box is measured up to 0.1 cm accuracy in its length, width and height. If length, width and height are respectively given by 3, 4, 5 meters, use differentials to estimate error in volume.  

Points (1)

- \( V = xyz = 60 \)
- \( dV = V_x \, dx + V_y \, dy + V_z \, dz \)
- \( dV = yz \, dx + xz \, dy + xy \, dz \)
- **ERROR:** \( dV = 20 \times 0.5 + 15 \times 0.5 + 12 \times 0.5 = 4.7 \)
Q7. Find if possible the limit \( \lim_{(x,y) \to (0,0)} \frac{x^6 - y^6}{x^3 - y^3} \) exists? \hspace{1cm} \text{Points (1.5)}

\checkmark \text{Write above as: } \lim_{(x,y) \to (0,0)} x^3 + y^3

\checkmark \text{Route 1: } \lim_{(x,0) \to (0,0)} x^3 + y^3 = 0

\checkmark \text{Route 2: } \lim_{(0,y) \to (0,0)} x^3 + y^3 = 0

\checkmark \text{Route 3: } \lim_{x (1,m) \to (0,0)} x^3 + y^3 = 0

\checkmark \text{Limit exists and equal to zero.}
Q8

(a) Find rectangular equation for the surface whose spherical equation is $\rho = \cos \phi$. Points (1)

(b) Describe in words surfaces whose equations are $\rho = 2$ and $\phi = \pi / 6$. Points (1)

(a)

✓ Note that: $x^2 + y^2 + z^2 = \rho^2$

✓ Substitute $\rho = \cos \phi$ in the above equation: $x^2 + y^2 + z^2 = \rho \cos \phi$

✓ Remember that: $z = \rho \cos \phi$

✓ Above equation becomes: $x^2 + y^2 + z^2 - z = 0$

✓ Complete square: $x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$

✓ The resulting equation is a sphere of radius $\frac{1}{2}$ and centre $(0, 0, \frac{1}{2})$

(b)

• First is a sphere of radius 2, with centre $(0,0,0)$ while second is right circular cone with vertex at origin.
Q8
(a): Show that the two planes given by \( x + 2y - z + 1 = 0 \) and \( 2x + 4y - 2z + 5 = 0 \) are parallel.

(b): Find distance between the planes given in part (a).

(a)

- Parallel planes have their normal parallel to each other.
- Normal vectors to the planes are: \( \vec{n}_1 = (1, 2, -1) \), \( \vec{n}_2 = (2, 4, -2) \)
- Dot product:
  \[
  \vec{n}_1 \cdot \vec{n}_2 = ||\vec{n}_1|| ||\vec{n}_2|| \cos \theta \Rightarrow 8 = \sqrt{6} \sqrt{24} \cos \theta \\
  \cos \theta = \frac{12}{2 \times 6} = 1 \Rightarrow \theta = 0
  \]
- Thus the two planes are parallel.

(b)

- From \( x + 2y - z + 1 = 0 \) we write:
  \[
  x = -1 - 2y + z \\
  y = y, \\
  z = z,
  \]
- Write above as:
  \[
  \begin{pmatrix}
  x \\
  y \\
  z
  \end{pmatrix} = \begin{pmatrix}
  -1 \\
  0 \\
  0
  \end{pmatrix} + y \begin{pmatrix}
  -2 \\
  0 \\
  0
  \end{pmatrix} + z \begin{pmatrix}
  1 \\
  1 \\
  1
  \end{pmatrix}
  \]
- The point on the plane is:
  \[
  \begin{pmatrix}
  x \\
  y \\
  z
  \end{pmatrix} = \begin{pmatrix}
  -1 \\
  0 \\
  0
  \end{pmatrix}
  \]
- Distance between the two parallel planes is:
  \[
  d = \frac{|-1 \times 2 + 0 \times 4 + 0 \times -2 + 5|}{\sqrt{4+16+4}} = \frac{1}{2 \sqrt{6}}
  \]