Q 1	Q 2	Q 3	Q 4	Total mark

Table 1: Table of Marks

King Fahd University of Petroleum and Minerals

Final Exam for Math 321

Semester 2, Academic year 2007-2008

Time allowed 2 hours minutes

Full Name: ....

ID Number: ....

Question 1

Given a function $f \in C[0,1]$. Let $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a partition of the interval $[0,1]$. Denote by S the piecewise linear interpolant of f corresponding to the partition.

a- Give a formula for S on each subinterval $[x_k, x_{k+1}]$ for $k = 0, \cdots, n-1$.

b- Assuming $f \in C^2[0,1]$, bound the error $f(x) - S(x)$.

Question 2

Suppose that $\{(x_k, y_k)\}_{k=0}^{n}$ are $n+1$ points, where $a = x_0 < x_1 < \cdots < x_n = b$. Let $S(x) \in C^2[a,b]$ be such that $S(x)|_{[x_{k-1},x_k]}$ is a cubic polynomial passing through (x_{k-1}, y_{k-1}) and (x_k, y_k) for $k = 1, \cdots, n$.

a- State the set of all constraints on $S(x)$.

b- Say whether $S(x)$ is unique or not? If not, what type of additional constraints can be assumed to guarantee the uniqueness of $S(x)$.

Question 3

The integral $\mathcal{I} = \int_a^b f(x)dx$ is approximated using the quadrature formula

$$Q[f] = \frac{3h}{2}[f(a+h) + f(a+2h)], \quad \text{where} \quad h = (b-a)/3.$$

a- Write down the Taylor series for $f(x)$ expanded about the midpoint c (of the interval $[a,b]$) up to the term in $(x-c)^2$. Integrate the terms of the Taylor series to show that

$$\mathcal{I} = 3hf(c) + \frac{9}{8}h^3 f''(c) + O(h^4).$$

b- Use the Taylor series for $f(x)$ expanded about c at $x = a+h$ and at $x = a+2h$ to show that

$$Q[f] = 3hf(c) + \frac{3}{8}h^3 f''(c) + O(h^4)$$

c- Derive the error $\mathcal{I} - Q[f]$, in this rule.

d- Subdivide the interval $[a,b]$ into 6 subintervals of equal length (say h) and derive the Composite quadrature rule for approximating $\mathcal{I}$ over 6 subintervals by applying the quadrature formula $Q[f]$ over successive sets of 3 subintervals.

e- Apply this composite rule to find an approximation to $\int_0^1 \sin x \, dx$. Calculate the exact value of the integral and then compute the absolute error in the approximation.

f- Compute the absolute error for approximating $\int_0^1 \sin x \, dx$ over 6 subinterval using the composite Simpson's rule over successive sets of 2 subintervals.

Question 4

Consider the initial value problem (IVP):

$$\frac{dy}{dt} = -ty, \quad t \in [0, 1] \text{ with } y(0) = 1.$$

a- Find the exact solution of the given IVP.

b- Use Euler method to solve the given IVP using step sizes of length $h = 1/3$, $1/6$ and $1/12$ respectively.

c- Compute the absolute error between the exact solution and the Euler solution for each choice of the step size h.

d- Let $|\mathrm{E}_k|$ be the absolute error between the exact solution and the Euler solution at the mesh gird t_k. Let $\mathrm{E}_h = \max\limits_{0 \le k \le n} |\mathrm{E}_k|$ where $n = 1/h$. Find E_h for $h = 1/3$, $1/6$ and $1/12$.

e- Demonstrate numerically the convergence order of the Euler method.