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Department of Mathematical Sciences
Exam 1 for Math 201 (081) (Solution)

Q.1: (12 pts) Find equations of all tangent lines to the parametric curve given by $x = t^5 - 4t^3,\ y = t^2$, at $(0, 4)$.

Sol: $x = 0$ if $t^3\left(t^2 - 4\right) = 0 \Rightarrow t = 0, \pm 2$ and $y = 4$ if $t^2 = 4 \Rightarrow t = \pm 2$.

The common values of t at which $x = 0$ and $y = 4$ are ± 2.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2}.$$

At $t = 2,\ m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{2(2)}{5(2)^4 - 12(2)^2} = \frac{1}{8}$ and equation of tangent line is $y = \frac{1}{8}x + 4$.

At $t = 2,\ m = \left.\frac{dy}{dx}\right|_{t=-2} = \frac{2(-2)}{5(-2)^4 - 12(-2)^2} = -\frac{1}{8}$ and equation of tangent line is $y = -\frac{1}{8}x + 4$.

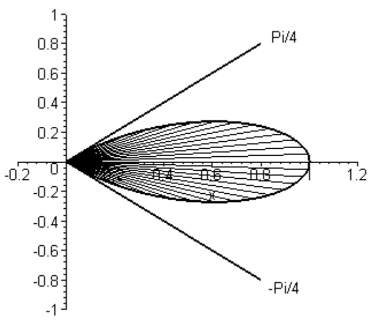
Q.2: (14 pts) Consider the polar curve $C:\ r = f(\theta) = \cos(2\theta),\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

(a) Sketch the curve.

(b) Setup the integral for the area enclosed by the curve.

(c) Setup the integral for the arc length of curve.

Sol: (a) The graph of $r = f(\theta) = \cos(2\theta)$

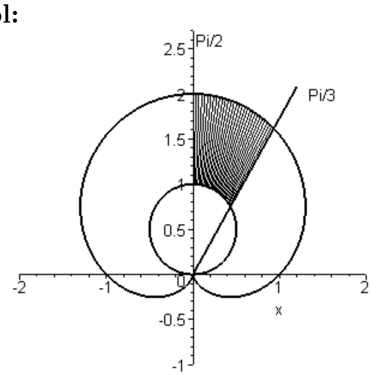


(b) $A = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta)\, d\theta.$

(c) $S = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos^2\theta + 4\sin^2(2\theta)}\, d\theta.$

Q.3: Find the area inside the curve $r = 1 + \sin(\theta)$ and outside the curve $r = \sin(\theta)$ when $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$.

Sol:



$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left((1 + \sin(\theta))^2 - \sin^2(\theta)\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(1+\sin^2(\theta)+2\sin(\theta)-\sin^2(\theta)\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+2\sin(\theta))\, d\theta = \frac{1}{12}\pi + \frac{1}{2}.$$

Q.4: (9 pts) Find an equation of the sphere if one of its diameters has end points at $A(1,4,-2)$ and $B(-7,1,2)$. What is the intersection of this sphere with the $xz-$plane ?

Sol: Center is $\left(\frac{1-7}{2}, \frac{4+1}{2}, \frac{-2+2}{2}\right) = \left(-3, \frac{5}{2}, 0\right)$ and radius is $r = \frac{1}{2}\sqrt{(-8)^2+(-3)^2+(4)^2} = \frac{1}{2}\sqrt{89}$.

Equation of the sphere is $(x+3)^2 + \left(y-\frac{5}{2}\right)^2 + (z-0)^2 = \frac{89}{4}$.

Intersection with $xz-$plane: Set $y=0$, then $(x+3)^2 + \left(0-\frac{5}{2}\right)^2 + (z-0)^2 = \frac{89}{4}$

$(x+3)^2 + z^2 = \frac{89}{4} - \frac{25}{4} = \frac{64}{4} = 16$, a circle in the $xz-$plane with radius 4 and center at $(-3,0,0)$.

Q.5: (9 pts) Find the distance from the point P(1,1,1) to the line passing through the points Q(2,-1,3) and R(5,0,1).

Sol: $\vec{QP} = \langle -1, 2, -2\rangle$ and $\vec{QR} = \langle 3, 1, -2\rangle$.

$$\vec{QR} \times \vec{QP} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -2 \\ -1 & 2 & -2 \end{vmatrix} = 2\hat{\imath} + 8\hat{\jmath} + 7\hat{k}.$$

$$D = \frac{|\vec{QR} \times \vec{QP}|}{|\vec{QR}|} = \frac{\left|2\hat{\imath}+8\hat{\jmath}+7\hat{k}\right|}{|3\hat{\imath}+\hat{\jmath}-2\hat{k}|} = \frac{\sqrt{4+64+49}}{\sqrt{9+1+4}} = \frac{\sqrt{117}}{\sqrt{14}} = \frac{3}{14}\sqrt{182}.$$

Q.6: (8 pts) Find the Cartesian equation of the curve whose polar equation is given by $r = \sec(\theta) - \csc(\theta)$.

Sol: $r = \frac{1}{\cos\theta} - \frac{1}{\sin\theta} = \frac{\sin\theta - \cos\theta}{\sin\theta\cos\theta}$

$r\sin\theta.\cos\theta = \sin\theta - \cos\theta$

$r\sin\theta.r\cos\theta = r\sin\theta - r\cos\theta$

$xy = y - x.$

$y = \frac{x}{1-x}.$

OR

$r = r = \frac{1}{\cos\theta} - \frac{1}{\sin\theta} = \frac{1}{\frac{x}{r}} - \frac{1}{\frac{y}{r}} = \frac{r}{x} - \frac{r}{y} = r\left(\frac{1}{x} - \frac{1}{y}\right)$

$1 = \frac{1}{x} - \frac{1}{y} \;\Rightarrow xy = y - x \;\Rightarrow y = \frac{x}{1-x}.$

OR

$x = r\cos\theta = (\sec\theta - \csc\theta)\cos\theta = 1 - \tan\theta$

$x = 1 - \frac{y}{x} \;\Rightarrow xy = y - x \;\Rightarrow y = \frac{x}{1-x}.$

Q.7: (6 pts) The graph of the curve represented by $x = 4\cos\theta$, and $y = 5\sin\theta$, is:

Sol: $x^2 = 16\cos^2\theta,\; y^2 = 25\sin^2\theta$

$\frac{x^2}{16} + \frac{y^2}{25} = \cos^2\theta + \sin^2\theta = 1$

This is an ellipse with center at the origin $(0,0)$ and intercepts $(\pm 4, 0)$ and $(0, \pm 5)$.

Q.8: (6 pts) What does the polar equation $r = \tan\theta\sec\theta$ represents ?

Sol: $r = \frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta} \;\Rightarrow r\cos^2\theta = \sin\theta \;\Rightarrow r^2\cos^2\theta = r\sin\theta$

$y^2 = x$, a parabola.

Q.9 : (6 pts) The vector that has the same direction as $\langle -3, 4, 1\rangle$ but has length 5 is:
Sol: $|\langle -3, 4, 1\rangle| = \sqrt{9+16+1} = \sqrt{26}$

A unit vector in the direction of $\langle -3, 4, 1\rangle$ is $\langle -\frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{1}{\sqrt{26}}\rangle$.

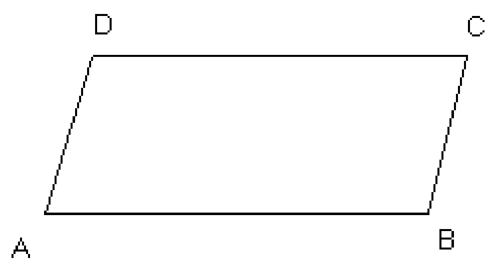
A vector of length 5 in the direction of $\langle -3, 4, 1\rangle$ is $\langle -\frac{15}{\sqrt{26}}, \frac{20}{\sqrt{26}}, \frac{5}{\sqrt{26}}\rangle$.

Q.10: (6 pts) The vector projection of $u = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ onto $v = 5\hat{\imath} - \hat{\jmath} + 2\hat{k}$ is given by:
Sol: $Proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \frac{5-2+6}{25+1+4}\langle 5, -1, 2\rangle = \frac{3}{10}\langle 5, -1, 2\rangle = \langle \frac{3}{2}, -\frac{3}{10}, \frac{3}{5}\rangle$.

Q.11: (6 pts) The Cartesian equation for the parametric equations $x = 9\sec t$ and $y = 8\tan t$ is:
Sol: $\frac{x^2}{9^2} = \sec^2 t$ and $\frac{y^2}{8^2} = \tan^2 t$

$\frac{x^2}{81} - \frac{y^2}{64} = \sec^2 t - \tan^2 t = 1$, a hyperbola.

Q.12: (6 pts) Let P be the parallelogram in with vertices $A = (1, -1, 2)$, $B = (2, 0, 1)$, $C = (3, 2, -1)$, $D = (2, 1, 0)$. The area of P is:
Sol: The vectors $\vec{AB} = \langle 1, 1, -1\rangle$ and $\vec{DC} = \langle 1, 1, -1\rangle$ shows that sides AB and DC are parallel with AB and AD as two adjecent sides.



Now $\vec{AD} = \langle 1, 2, -2\rangle$ and $\vec{AB}\times\vec{AD} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{vmatrix} = \hat{\imath} + \hat{k}$

Area of P is $\mathbf{A} = \left|\vec{AB}\times\vec{AD}\right| = \sqrt{2}$.