# King Fahd University of Petroleum and Minerals
## Department of Mathematics and Statistics
### Exam 2 for Math 201 (081) (Solution)

---

**Q.1:** ($a$) Find the point at which these lines intersect:

$x = 1 + t,\ y = 1 - t,\ z = 2t$ and $x = 2 - s,\ y = s,\ z = 2$.

($b$) Determine an equation of the plane that contains these lines.

**Sol:** ($a$) Solving the system

$$\begin{aligned} 1 + t &= 2 - s \\ 1 - t &= s \end{aligned}$$

we get $t = 1$ and $s = 2$. Putting $t = 1$ in the first equation or $s = 2$ in the second equation, we get the intersection point $(2, 0, 2)$.

($b$) Direction vectors of these lines are $\vec{v}_1 = \langle 1, -1, 2 \rangle$ and $\vec{v}_2 = \langle -1, 1, 0 \rangle$. Since the required plane is passing through these lines. therefore these vectors are parallel to the plane. Hence the vector normal to the plane is $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = -2i - 2j + 0k.$

Equation of the plane is $-2(x-2) - 2(y-0) + 0(z-2) = 0 \Rightarrow x + y - 2 = 0.$

**Q.2:** Consider the surface $\dfrac{z}{4} = \sqrt{x^2 + y^2}$.

($a$) Describe the traces along the $z-$axis (parallel to $xy-$plane).

($b$) Describe the traces along the $x-$axis (parallel to $yz-$plane).

($c$) Identify and sketch the surface.

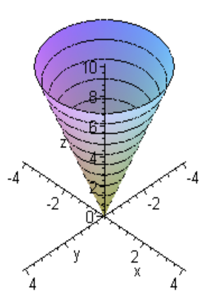
**Sol:** ($a$) For traces along $z-$axis, put $z = k$ with $k \geq 0$,

then $\sqrt{x^2 + y^2} = \dfrac{k}{4}$ or $x^2 + y^2 = \left(\dfrac{k}{4}\right)^2$ is a family of circles.

($b$) For traces along $x-$axis, put $x = k$,

then $\sqrt{k^2 + y^2} = \dfrac{z}{4}$ or $k^2 = \left(\dfrac{z}{4}\right)^2 - y^2$ a family of hyperbolas (upper half in the planes parallel to $yz-$planes) with $z \geq 0$. For $k = 0$, it gives straight lines $z = 4|y|$ passing through the origin $(0, 0, 0)$.

($c$) The surface is a cone $\sqrt{x^2 + y^2} = \dfrac{z}{4}$



**Q.3:** Let $f(x,y) = \ln\left(36 - 4x^2 - 9y^2\right)$.

$(a)$ Find and sketch the domain of $f$

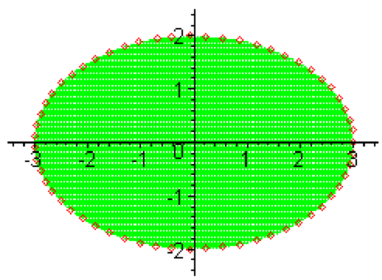
$(b)$ Find the range of $f$

**Sol:** $(a)$ $36 - 4x^2 - 9y^2 > 0$

$4x^2 + 9y^2 < 36$

$\frac{x^2}{9} + \frac{y^2}{4} < 1$

Therefore the domain is all points inside an open ellipse $\left\{(x,y) \mid \frac{x^2}{9} + \frac{y^2}{4} < 1\right\}$




$(b)$ Range of $f$ is $(-\infty, \ln 36]$

**Q.4:** If $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$, does the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist? Justify your answer.

**Sol:** Let $(x,y) \to (0,0)$ through $y - axis$, that is $x = 0$ and $y \to 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{y\to 0} \frac{0.y}{0 + y^2} = 0$$

Now let $(x,y) \to (0,0)$ through $x - axis$, that is $y = 0$ and $x \to 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^2.0}{x^4 + 0} = 0.$$

Now let $(x,y) \to (0,0)$ through $y = x^2$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^2.x^2}{x^4 + x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since these limits are different, therefore limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

**Q.5:** Suppose over certain region of space the electrical potential $V$ is given by $V(x,y,x) = 3xy^2 - y^2 + xyz$.
$(a)$ Compute the rate of change of the petential at $A(1,1,-1)$ in the direction of $\vec{u} = 2\hat{\imath} + \hat{\jmath} - 3\hat{k}$.
$(b)$ In which direction does $V$ changes most rapidly?
$(c)$ What is the maximum rate of change at $A$?

**Sol:** Gradient vector is $\nabla V(x,y,z) = \langle 3y^2 + yz, 6xy - 2y + xz, xy\rangle$

At $A(1,1,-1)$ $\nabla V(1,1,-1) = \langle 2,3,1\rangle$

The unit vector in the direction of $\vec{u}$ is $\hat{u} = \dfrac{\vec{u}}{\|\vec{u}\|} = \langle \dfrac{2}{\sqrt{14}}, \dfrac{1}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}\rangle$.

Rate of change of $V$ in the direction of $\vec{u}$ is

$$D_{\vec{u}}V(1,1,-1) = \nabla V(1,1,-1) \cdot \frac{\vec{u}}{\|\vec{u}\|} = \langle 2,3,1\rangle \cdot \langle \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\rangle$$
$$= \frac{4+3-3}{\sqrt{14}} = \frac{4}{\sqrt{14}}.$$

$(b)$ $V$ changes most rapidly at $A(1,1,-1)$ in the direction of the gradient vector $\nabla V(1,1,-1) = \langle 2,3,1\rangle$.

$(c)$ The maximum rate of change at $A(1,1,-1)$ is $\|\nabla V(1,1,-1)\| = \sqrt{4+9+1} = \sqrt{14}$.

**Q.6:** Find the maximum and minimum values of $f(x,y) = xy - x^3y^2$ over the region $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$.

**Sol:** Find critical points inside the region $R$.

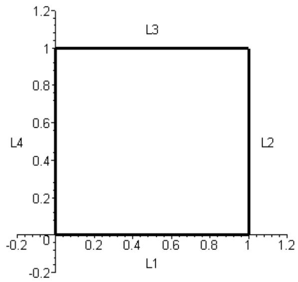
$f_x(x,y) = y - 3x^2y = y(1-3x^2y)$, $f_y(x,y) = x - 2x^3y = x(1-2x^2y)$.

$f_x(x,y) = 0$, $f_y(x,y) = 0 \Rightarrow x = 0, y = 0$, OR $3x^2y = 1$ and $2x^2y = 1$.

But the system $3x^2y = 1$ and $2x^2y = 1$ has no solution inside the region $R$.

Thus the function $f(x,y) = xy - x^3y^2$ has no critical point inside $R$.

The boundary of $R$ is a square which consists of line segments: $L_1 = \{(x,y) \mid y = 0,\ 0 \le x \le 1\}$, $L_2 = \{(x,y) \mid x = 1,\ 0 \le y \le 1\}$, $L_3 = \{(x,y) \mid y = 1,\ 0 \le x \le 1\}$, $L_4 = \{(x,y) \mid x = 0,\ 0 \le y \le 1\}$




On $L_1$, $g(x) = f(x,0) = 0$ for all $0 \le x \le 1$.

On $L_2$, $g(y) = f(1,y) = y - y^2$, $0 \le y \le 1$ and $g(0) = f(1,0) = 0$, $g(1) = f(1,1) = 0$.

For the critical points on $L_2$, $g'(y) = 1 - 2y = 0 \Rightarrow y = \dfrac{1}{2}$ and $g\left(\frac{1}{2}\right) = f(1, \frac{1}{2}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{2}$.

On $L_3$, $g(x) = f(x,1) = x - x^3$, $0 \le x \le 1$ and $g(0) = f(0,1) = 0$, $g(1) = f(1,1) = 0$.

For the critical points on $L_3$, $g'(x) = 1 - 3x^2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$. But $-\frac{1}{\sqrt{3}} \notin R$.

and $g\left(\frac{1}{\sqrt{3}}\right) = f\left(\frac{1}{\sqrt{3}}, 1\right) = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}$.

On $L_4$, $g(y) = f(0,y) = 0$ for all $0 \le y \le 1$.

Thus maximum value of $f$ is $\frac{2}{3\sqrt{3}}$ and minimum value of $f$ is 0.

**Q.7:** Let $f(x,y) = \sin\left(\frac{x}{y}\right) + e^{\frac{x}{y}} + 3x$. Then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is equal to:

(a) 0

(b) $\sin\left(\dfrac{x}{y}\right) - e^{\frac{x}{y}} + 3$

(c) $\cos\left(\dfrac{x}{y}\right) + e^{\frac{x}{y}}$

(d) $\cos\left(\dfrac{x}{y}\right) - e^{\frac{x}{y}}$

(e) $3x$ −−−−−−→Correct Answer

**Sol:** $\dfrac{\partial f}{\partial x} = \cos\left(\frac{x}{y}\right).\frac{1}{y} + e^{\frac{x}{y}}.\frac{1}{y} + 3$ and $x\dfrac{\partial f}{\partial x} = \frac{x}{y}\cos\left(\frac{x}{y}\right) + \frac{x}{y}e^{\frac{x}{y}} + 3x$.

$\dfrac{\partial f}{\partial y} = \cos\left(\frac{x}{y}\right).\frac{-x}{y^2} + e^{\frac{x}{y}}.\frac{-x}{y^2}$ and $y\dfrac{\partial f}{\partial y} = \frac{-x}{y}\cos\left(\frac{x}{y}\right) - \frac{x}{y}e^{\frac{x}{y}}$.

$x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 3x$.

**Q.8:** The distance between the point $(2,-3,4)$ and the plane $x + 2y + 2z - 13 = 0$ is equal to:

(a) 7

(b) 3 −−−−−−→Correct Answer

(c) 1

(d) 4

(e) 2

**Sol:** $$d = \frac{|2 + 2(-3) + 2(4) - 13|}{\sqrt{1+4+4}} = 3.$$

**Q.9 :** Find $\left.\dfrac{\partial z}{\partial x}\right|_{\left(\frac{\pi}{4},1,1\right)}$ when $x - z + 1 = \arctan(yz)$.

$(a)$ 0
$(b)$ 1
$(c)$ $\dfrac{1}{2}$
$(d)$ $\dfrac{2}{3}$ $------\rightarrow$Correct Answer
$(e)$ $-\dfrac{2}{3}$

**Sol:** Here $F(x,y,z) = x - z + 1 - \arctan(yz) = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1}{-1-\frac{y}{1+(yz)^2}} = \frac{1}{\frac{1+(yz)^2+y}{1+(yz)^2}} = \frac{1+(yz)^2}{1+(yz)^2+y}$$

$$\left.\frac{\partial z}{\partial x}\right|_{\left(\frac{\pi}{4},1,1\right)} = \frac{1+(1)^2}{1+(1)^2+1} = \tfrac{2}{3}.$$

**Q.10:** If $\left(0, -2\sqrt{3}, -2\right)$ is a point in rectangular coordinates and $\rho, \theta, \phi$ are its spherical coordinates, then $\rho + \tan\phi + \csc\theta$ is equal to:

$(a)$ $3 - \sqrt{3}$ $------\rightarrow$Correct Answer
$(b)$ $3 + \sqrt{3}$
$(c)$ $5 + \sqrt{3}$
$(d)$ $5 - \sqrt{3}$
$(e)$ None of these

**Sol:**

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0+12+4} = 4$$

$$\cos\phi = \frac{z}{\rho} = \frac{-2}{4} = \frac{-1}{2} \Rightarrow \phi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho\sin\phi} = \frac{0}{4\sin\frac{2\pi}{3}} = 0 \Rightarrow \theta = \frac{3\pi}{2},\ (\theta \neq \frac{\pi}{2} \text{ since } y = -2\sqrt{3} < 0)$$

$$\rho + \tan\phi + \csc\theta = 4 + \tan\frac{2\pi}{3} + \csc\frac{3\pi}{2} = 4 + \left(-\sqrt{3}\right) + (-1) = 3 - \sqrt{3}.$$

**Q.11:** If $(x, y)$ changes from $(2, -1)$ to $(1.96, -0.95)$ in the function $z = x^2 - xy + 3y^2$, then the value of the differential $dz$ is:

$(a)$ 1.6
$(b)$ 0.45
$(c)$ $-0.60$ $------\rightarrow$Correct Answer
$(d)$ 0.03
$(e)$ $-1.5$

**Sol:** Here $x = 2,\ y = -1,\ \Delta x = dx = -0.04$, and $\Delta y = dy = 0.05$.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x - y)\,dx + (-x + 6y)\,dy$$
$$= (4+1)(-0.04) + (-2-6)(0.05) = -0.2 - 0.4 = -0.6.$$

**Q.12:** Suppose that $w = r^2\cos(2\theta)$, where $r$ and $\theta$ are the polar coordinates, then $\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y}$ is equal to:

$(a)$ $3(x-y)$
$(b)$ $2(x-y)$ $------\rightarrow$Correct Answer
$(c)$ $3(x+y)$
$(d)$ $2x - 3y$
$(e)$ $3x + 2y$

**Sol:** $w = r^2\left(\cos^2\theta - \sin^2\theta\right) = x^2 - y^2$ and $\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} = 2x - 2y = 2(x-y)$.