Q.1: (12 pts) Find equations of all tangent lines to the parametric curve given by \( x = t^5 - 4t^3 \), \( y = t^2 \), at \((0, 4)\).

Sol: \( x = 0 \) if \( t^3 (t^2 - 4) = 0 \) \( \Rightarrow t = 0, \pm 2 \) and \( y = 4 \) if \( t^2 = 4 \) \( \Rightarrow t = \pm 2 \).

The common values of \( t \) at which \( x = 0 \) and \( y = 4 \) are \( \pm 2 \).

\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{2t}{5t^4 - 12t^2}.
\]

At \( t = 2 \), \( m = \frac{dy}{dx} \bigg|_{t=2} = \frac{2 (2)}{5 (2)^4 - 12 (2)^2} = \frac{1}{8} \) and equation of tangent line is \( y = \frac{1}{8} x + 4 \).

At \( t = -2 \), \( m = \frac{dy}{dx} \bigg|_{t=-2} = \frac{2 (-2)}{5 (-2)^4 - 12 (-2)^2} = -\frac{1}{8} \) and equation of tangent line is \( y = -\frac{1}{8} x + 4 \).

Q.2: (14 pts) Consider the polar curve \( r = f(\theta) = \cos(2\theta) \), \( -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \)

(a) Sketch the curve.

(b) Setup the integral for the area enclosed by the curve.

(c) Setup the integral for the arc length of curve.

Sol: (a) The graph of \( r = f(\theta) = \cos(2\theta) \)

\[
A = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) \, d\theta.
\]

(b) \( S = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos^2 \theta + 4 \sin^2(2\theta)} d\theta.\)

Q.3: Find the area inside the curve \( r = 1 + \sin(\theta) \) and outside the curve \( r = \sin(\theta) \) when \( \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \).

Sol:
\[ A = \frac{1}{2} \int_{\pi/3}^{\pi} ((1 + \sin(\theta))^2 - \sin^2(\theta)) \, d\theta \]
\[ = \frac{1}{2} \int_{\pi/3}^{\pi} (1 + \sin^2(\theta) + 2 \sin(\theta) - \sin^2(\theta)) \, d\theta \]
\[ = \frac{1}{2} \int_{\pi/3}^{\pi} (1 + 2 \sin(\theta)) \, d\theta = \frac{1}{12} \pi + \frac{1}{2}. \]

**Q.4:** (9 pts) Find an equation of the sphere if one of its diameters has end points at \( Q = (1, 4, -2) \) and \( B = (-7, 1, 2) \). What is the intersection of this sphere with the \( xy-\)plane?

**Sol:** Center is \( \left( \frac{1 - 7}{2}, \frac{4 + 1}{2}, \frac{-2 + 2}{2} \right) = \left( -3, \frac{5}{2}, 0 \right) \) and radius is \( r = \frac{1}{2} \sqrt{(-8)^2 + (-3)^2 + (4)^2} = \frac{1}{2} \sqrt{89}. \)

Equation of the sphere is \( (x + 3)^2 + \left( y - \frac{5}{2} \right)^2 + (z - 0)^2 = \frac{89}{4}. \)

Intersection with \( xy-\)plane: Set \( y = 0 \), then \( (x + 3)^2 + \left( 0 - \frac{5}{2} \right)^2 + (z - 0)^2 = \frac{89}{4} \)
\( (x + 3)^2 + z^2 = \frac{89}{4} - \frac{25}{4} = \frac{64}{4} = 16, \) a circle in the \( xy-\)plane with radius 4 and center at \((-3, 0, 0).\)

**Q.5:** (9 pts) Find the distance from the point \( P = (1, 1, 1) \) to the line passing through the points \( Q = (2, -1, 3) \) and \( R = (5, 0, 1). \)

**Sol:** \( Q \bar{P} = (-1, 2, -2) \) and \( Q \bar{R} = (3, 1, -2). \)
\[ Q \bar{R} \times Q \bar{P} = \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ -1 & 2 & -2 \end{vmatrix} = 2i + 8j + 7k. \]
\[ D = \frac{|Q \bar{R} \times Q \bar{P}|}{|Q \bar{R}|} = \frac{|2i + 8j + 7k|}{\sqrt{3^2 + 8^2 + (-2)^2}} = \frac{\sqrt{4 + 64 + 49}}{\sqrt{9 + 1 + 4}} = \frac{\sqrt{117}}{\sqrt{14}} = 3 \sqrt{\frac{117}{14}}. \]

**Q.6:** (8 pts) Find the Cartesian equation of the curve whose polar equation is given by \( r = \sec(\theta) - \csc(\theta). \)

**Sol:** \( r = \frac{1}{\cos \theta} - \frac{1}{\sin \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta \cos \theta} \)
\( r \sin \theta \cdot \cos \theta = \sin \theta - \cos \theta \)
\( x = r \sin \theta \cos \theta = r \sin \theta = r \cos \theta \)
\( y = x \left( 1 - x \right). \)

OR
\[ r = r \left( \frac{1}{\cos \theta} - \frac{1}{\sin \theta} \right) = \frac{x}{r} - \frac{y}{r} = \frac{r - r}{x - y} = r \left( \frac{1 - \frac{1}{x}}{x - y} \right) \]
\[ 1 = \frac{1}{x} - \frac{1}{y} \Rightarrow xy = y - x \Rightarrow y = \frac{x}{1 - x}. \]

OR
\[ x = r \cos \theta = (\sec \theta - \csc \theta) \cos \theta = 1 - \tan \theta \]
\[ y = \frac{y}{x} \Rightarrow xy = y - x \Rightarrow y = \frac{x}{1 - x}. \]

**Q.7:** (6 pts) The graph of the curve represented by \( x = 4 \cos \theta, \) and \( y = 5 \sin \theta, \) is:

**Sol:** \[ x^2 = 16 \cos^2 \theta, \quad y^2 = 25 \sin^2 \theta \]
\[ x^2 + y^2 = \cos^2 \theta + \sin^2 \theta = 1 \]
This is an ellipse with center at the origin \((0, 0)\) and intercepts \((\pm 4, 0)\) and \((0, \pm 5)\).
Q.8: (6 pts) What does the polar equation \( r = \tan \theta \sec \theta \) represents?

Sol: 
\[
\begin{align*}
\frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta \cos \theta} &= \frac{\sin \theta}{\cos \theta} \rightarrow \cos \theta \sin \theta = r \cos^2 \theta \\
y^2 &= x, \text{ a parabola.}
\end{align*}
\]

Q.9: (6 pts) The vector that has the same direction as \( {-3i + 4j + k} \) but has length 5 is:

Sol: 
\[
\begin{align*}
\text{A unit vector in the direction of } &{-3i + 4j + k} \\
\text{is } &\left\langle -\frac{3}{5}, \frac{4}{5}, \frac{1}{5} \right\rangle.
\end{align*}
\]

A vector of length 5 in the direction of \( {-3i + 4j + k} \) is:
\[
\begin{align*}
&3i + 4j + \frac{1}{5}k.
\end{align*}
\]

Q.10: (6 pts) The vector projection of \( u = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \) onto \( v = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \) is given by:

Sol: 
\[
\text{Proj}_v \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2} \mathbf{v} = \frac{5 - 2 + 6}{25 + 1 + 4} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \frac{3}{10} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -\frac{3}{10} \\ \frac{3}{5} \end{pmatrix}.
\]

Q.11: (6 pts) The Cartesian equation for the parametric equations \( x = 9 \sec t \) and \( y = 8 \tan t \) is:

Sol: 
\[
\begin{align*}
\frac{x^2}{81} - \frac{y^2}{64} &= \sec^2 t - \tan^2 t = 1, \text{ a hyperbola.}
\end{align*}
\]

Q.12: (6 pts) Let \( P \) be the parallelogram in with vertices \( A = (1, -1, 2), B = (2, 0, 1), C = (3, 2, -1), \) \( D = (2, 1, 0). \) The area of \( P \) is:

Sol: 
The vectors \( \overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \) and \( \overrightarrow{DC} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \) shows that sides AB and DC are parallel with AB and AD as two adjacent sides.

Now \( \overrightarrow{AD} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \) and \( \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 1 & 2 & -1 \end{vmatrix} = i + k \)

Area of \( P \) is \( A = \left| \overrightarrow{AB} \times \overrightarrow{AD} \right| = \sqrt{2}. \)