1. Write clearly.

2. Show all your steps.

3. No credit will be given to wrong steps.

4. Do not do messy work.

5. Calculators and mobile phones are NOT allowed in this exam.
1.

(a) Find a matrix $P$ that diagonalizes the matrix

$$A = \begin{bmatrix}
\frac{3}{2} & \frac{i}{2} & 0 \\
-\frac{i}{2} & \frac{2-i}{2} & 0 \\
0 & 0 & 1
\end{bmatrix}$$

(b) Let $A = BDB^{-1}$, where $B$ is a non-singular matrix and $D$ is a diagonal matrix. Prove that $A$ and $D$ have the same eigenvalues.
2. Let the vector field \( \vec{F} = ye^{xy} \cos(z) \hat{i} + xe^{xy} \cos(z) \hat{j} - e^{xy} \sin(z) \hat{k} \).

(a) Show that the vector field \( \vec{F} \) is conservative.
(b) Find the potential function.
(c) Use part (b) to evaluate

\[
\int_{C} \vec{F} \cdot d\vec{r}
\]

from the point \((0, 0, 0)\) to \((-1, 2, \pi)\).
3. Solve the following:

(a) Find the cube roots of the $-1 + i$

(b) Find all roots of the algebraic equation

\[ z^4 + 6iz^2 + 16 = 0 \]

(c) Find $z$ where $z = (1 + i)^{(1-i)}$. 
4. Where the Cauchy-Riemann equations satisfied for the following function?

\[ f(z) = z + \frac{1}{z}. \]
5. Integrate the function \( f(z) = z^2 \) along two paths from \( z = 0 \) to \( z = 2 + i \). (See the attached figure).
6. Use Cauchy integral formula to find the value of

\[ I = \int_{|z|=2} \frac{e^z}{(z - 1)^2(z - 3)} \, dz \]
7. Find the Laurent expansion for

\[ f(z) = \frac{z}{(z - 1)(z - 3)}, \]

about the point \( z = 1 \).
8. Use Green’s theorem to evaluate
\[ \oint_C y^2 \, dx + xy \, dy, \]
around the curve $C$ defined by the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$, oriented counter clockwise.
9. Use the divergence theorem to evaluate \( \iint_S \vec{F} \cdot d\vec{S} \), where \( \vec{F} = xy\vec{i} - \frac{1}{2}y^2 \vec{j} + z\vec{k} \), and the surface consists of the three surfaces \( z = 4 - 3x^2 - 3y^2 \), \( 1 \leq z \leq 4 \), on the top, \( x^2 + y^2 = 1 \), \( 0 \leq z \leq 1 \) on the sides and \( z = 0 \) on the bottom.

**Hint:** Use the polar coordinates.
10. Use the residue theorem to evaluate
\[ \frac{1}{2\pi i} \oint_C \frac{e^{tz}}{z^2(z^2 + 2z + 2)} \, dz, \]

where \( C \) includes all of the singularities and is in the positive sense.
11. Use the residue theorem to evaluate
\[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1} \]
12. Define the function $\phi$ as follows:

$$\phi(x, y, z) = \sin(yz) + \ln(x^2)$$

(a) Compute the directional derivative of $\phi$ at the point $(1, 1, \pi)$ in the direction of the vector $(1, 1, -1)$.

(b) What is the direction of maximum change of $\phi$ at the point $(1, 1, \pi)$? What is the magnitude of this change.