King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101- Calculus I
Exam I
2008-2009 (082)

Monday, March 30, 2009

Name: KEY

ID Number:

Section Number: Serial Number:

Instructions:
1. Write neatly and legibly. You may lose points for messy work.
2. Show all your work. No points for answers without justification.
3. Calculators and Mobiles are not allowed.
4. Make sure that you have 9 different problems (5 pages + cover page)

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1. (a) [3 points] Write the following statement as a limit:

"$f(x)$ increases without bound as $x$ approaches $a$ from the left."

$$\lim_{{x \to a^-}} f(x) = +\infty$$

(b) [4 points] TRUE or FALSE: "If $f$ has a domain $[0, +\infty)$ and has no horizontal asymptote, then $\lim_{{x \to +\infty}} f(x) = +\infty$ or $\lim_{{x \to +\infty}} f(x) = -\infty$".

[If TRUE, state the reason. If FALSE, illustrate graphically].

   1. **False**: Take a periodic function $f$.

   2. [Graph of a periodic function]

(c) [7 points] Sketch the graph of a function $f$ that satisfies the following conditions:

   i. $f(-1) = 3$
   ii. $\lim_{{x \to -1^-}} f(x) = 4$
   iii. $\lim_{{x \to -1^+}} f(x) = -\infty$
   iv. $f(3)$ is undefined
   v. $\lim_{{x \to 3}} f(x) = 2$
   vi. $\lim_{{x \to +\infty}} f(x) = +\infty$
   vii. $\lim_{{x \to -\infty}} f(x) = 0$

   **Other graphs are possible**
2. Find the limit if it exists.

(a) [6 points] \(\lim_{x \to -4} \frac{x^3 - 16x}{x + 4} = \lim_{x \to -4} \frac{x(x-4)(x+4)}{x-4} = \lim_{x \to -4} x(x-4) = 32\)

(b) [6 points] \(\lim_{x \to 12} \frac{|12 - x|}{x - 12}\)

\[\lim_{x \to 12^-} \frac{|12 - x|}{x - 12} = \lim_{x \to 12^-} \frac{12 - x}{x - 12} = \lim_{x \to 12^-} -1 = -1\]

\[\lim_{x \to 12^+} \frac{|12 - x|}{x - 12} = \lim_{x \to 12^+} \frac{12 - x}{x - 12} = \lim_{x \to 12^+} 1 = 1\]

Since \(\lim_{x \to 12^-} \frac{|12 - x|}{x - 12} \neq \lim_{x \to 12^+} \frac{|12 - x|}{x - 12}\), then \(\lim_{x \to 12} \frac{|12 - x|}{x - 12}\) does not exist.

(c) [6 points] \(\lim_{x \to 3} g(x), \text{ where } 2x - 1 \leq g(x) \leq x^2 - 5x + 11\)

\[\lim_{x \to 3} \frac{2x - 1}{x - 3} = 5\]

\[\lim_{x \to 3} \frac{x^2 - 5x + 11}{x - 3} = 5\]

Then by the Squeeze Theorem, \(\lim_{x \to 3} g(x) = 5\)

(d) [6 points] \(\lim_{x \to 6^+} \tan^{-1}(\ln(x - 6))\)

\[\text{as } x \to 6^+, \ln(x - 6) \to -\infty \Rightarrow \tan^{-1}(\ln(x - 6)) \to -\frac{\pi}{2}\]

Thus, \(\lim_{x \to 6^+} \tan^{-1}(\ln(x - 6)) = -\frac{\pi}{2}\)
3. [8 points] Using the $\epsilon, \delta$ definition of limit, prove that $\lim_{x \to 1} \left( -1 + \frac{3}{2}x \right) = \frac{1}{2}$.

Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

$$|(-1 + \frac{3}{2}x) - \frac{1}{2}| < \varepsilon \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

But $|(-1 + \frac{3}{2}x) - \frac{1}{2}| = |\frac{3}{2}x - \frac{3}{2}| = \frac{3}{2} |x - 1|$. Thus, we want

$$\frac{3}{2} |x - 1| < \varepsilon \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

That is,

$$|x - 1| < \frac{2\varepsilon}{3} \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

Thus, we may choose $\delta = \frac{2\varepsilon}{3}$.

Note: No need to check that $\delta = \frac{2\varepsilon}{3}$ works.

4. [8 points] Let $f(x) = \begin{cases} \sqrt{x+2} & \text{if} \ -2 \leq x \leq 2 \\ x^3 - 2x & \text{if} \ x > 2 \end{cases}$ Is $f$ continuous at $x = 2$. If not, what kind of discontinuity does $f$ have at $x = 2$. Justify your answers.

- $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^3 - 2x = 8 - 4 = 4$

- $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sqrt{x+2} = \sqrt{4} = 2$

Since $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$, then $\lim_{x \to 2} f(x)$ does not exist.

- $\lim_{x \to 2^+} f(x)$ exists, $\lim_{x \to 2^-} f(x)$ exists, & $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$, hence $f$ is not continuous at $x = 2$.

Then $f$ has a jump discontinuity at $x = 2$.

5. [6 points] Where is the function $f(x) = \frac{1}{3 - \sqrt{x}}$ continuous? Is it continuous in its domain?

- Because of the term $\sqrt{x}$, we must have $x \geq 0$.

- We also must have $3 - \sqrt{x} \neq 0$:

  $3 - \sqrt{x} = 0 \Rightarrow \sqrt{x} = 3 \Rightarrow x = 9$.

  So we must have $x \neq 9$.

- $f$ is continuous in $[0, 9) \cup (9, +\infty)$.
6. [8 points] Show that the equation $e^{-x} = 2 - x$ has a root in the interval $(1, 2)$.

1. **Apply the Intermediate Value Theorem** by letting 
   \[ f(x) = e^{-x} + x - 2, \quad [a, b] = (1, 2), \quad N = 0 \]
2. \( f \) is **continuous** on \((1, 2)\)
3. \( f(1) = e^{-1} + 1 - 2 = \frac{1}{e} - 1 = \frac{1 - e}{e} < 0 \)
4. \( f(2) = e^{-2} + 2 - 2 = e^{-2} > 0 \)

So \( N = 0 \) is between \( f(1) \) and \( f(2) \). Then by the IVT, there is a number \( c \) in \((1, 2)\) such that \( f(c) = 0 \), that is \( e^{-c} + c - 2 = 0 \) or \( e^{-c} = 2 - c \).

7. (a) [8 points] Find \( \lim_{x \to +\infty} (\sqrt{x^2 + 1} - x) \).

\[
\lim_{x \to +\infty} \frac{\sqrt{x^2 + 1} - x}{1} = \frac{x^2 + 1 + x}{\sqrt{x^2 + 1} + x} \]

\[
= \lim_{x \to +\infty} \frac{1}{\sqrt{x^2 + 1} + x} \]

\[
= \lim_{x \to +\infty} \frac{1}{\sqrt{x^2 + 1} + x} \]

\[
= \lim_{x \to +\infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}} + 1} \]

\[
= \frac{0}{1 + 1} = 0 \]

(b) [8 points] Find the horizontal asymptotes of \( f(x) = e^{x^2-x^2} \).

We find \( \lim_{x \to +\infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).

Let \( u = x - x^2 \). Then
\[
\lim_{x \to +\infty} u = \lim_{x \to +\infty} x - x^2 = \lim_{x \to +\infty} x(\frac{1}{x} - 1) = +\infty (0-1) = -\infty \]

So \( \lim_{x \to +\infty} f(x) = \lim_{u \to -\infty} e^u = 0 \)

\[
\lim_{x \to -\infty} f(x) = \lim_{u \to +\infty} e^u = 0 \]

H.A. \( Y = 0 \)
8. [8 points] Find an equation of the tangent line to the curve \( y = \frac{1}{x^2 - x} \) at the point \((2, \frac{1}{2})\). [You must use limits]

\[
\begin{align*}
\text{Slope} &= m = \lim_{x \to 2} \frac{\frac{f(x) - f(2)}{x - 2}} \quad \text{\( \boxed{2} \)} \\
&= \lim_{x \to 2} \frac{\frac{1}{x^2 - x} - \frac{1}{2}}{x - 2} \\
&= \lim_{x \to 2} \frac{\frac{2 - x^2 + x}{2(x^2 - x)(x - 2)}}{x - 2} = \lim_{x \to 2} \frac{-(x-2)(x+1)}{2x(x-1)(x-2)} \\
&= \lim_{x \to 2} \frac{-3}{4} = \frac{-3}{4} \quad \boxed{1}
\end{align*}
\]

An equation for the tangent line is

\[
y - \frac{1}{2} = -\frac{3}{4} (x - 2) \quad \boxed{3}
\]

\[
\Rightarrow \quad y = -\frac{3}{4} x + 2 \quad \boxed{5}
\]

9. The displacement (in meters) of a particle moving in a straight line is given by the equation \( s(t) = 3t^2 - 4t + 1 \), where \( t \) is measured in seconds.

(a) [2 points] Find the average velocity over the time interval \([0, 3]\).

\[
\nu_{\text{ave}} = \frac{s(3) - s(0)}{3 - 0} \quad \boxed{1}
\]

\[
= \frac{16 - 1}{3} = \frac{15}{3} = 5 \text{ m/s} \quad \boxed{5}
\]

(b) [6 points] Use limits to find the instantaneous velocity when \( t = 2 \).

\[
\nu(t) = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} \quad \boxed{2}
\]

\[
= \lim_{h \to 0} \frac{3(2+h)^2 - 4(2+h) + 1 - 5}{h} \quad 2
\]

\[
= \lim_{h \to 0} \frac{3(4+4h+h^2) - 8 - 4h}{h} \quad \boxed{2}
\]

\[
= \lim_{h \to 0} \frac{8 + 3h}{h} \quad \boxed{2}
\]

\[
= 8 \text{ m/s} \quad \boxed{2}
\]