Q.1: (a) (2-points) Consider the differential equation
\[ \frac{dy}{dx} = y^2 \] (A)
Verify that \( y = \frac{-1}{x + c} \) is a one parameter family of solutions. (A).

Sol: \( y = \frac{-1}{x + c} \Rightarrow \frac{dy}{dx} = -1 \left( -1 \right) \left( x + c \right)^2 = \frac{1}{(x + c)^2} = y^2. \)

(b) (2-points) Find solutions of equation (A) which satisfies the initial condition \( y(0) = 2 \).

Sol: \( y(0) = 2 \Rightarrow 2 = \frac{-1}{0 + c} \Rightarrow c = -\frac{1}{2} \) and \( y = \frac{-1}{x - \frac{1}{2}} = \frac{-2}{x - 1}. \)

(c) (2-points) Find, if any, singular solutions of (A) and determine the largest interval of existence of each singular solution obtained.

Sol: Singular solutions are given by \( y^2 = 0 \Rightarrow y = 0 \) and \( \frac{dy}{dx} = 0. \)
The largest interval of existence is \( (-\infty, \infty) \).

Q.2: (a) (3-points) Given that \( x(t) = c_1 \cos(wt) + c_2 \sin(wt) \) is the general solution of the differential equation
\[ \ddot{x} + wx = 0 \quad \text{on} \quad (-\infty, \infty), \quad w \neq 0. \]

Find all solutions which satisfy the initial conditions \( x(0) = 0 \) and \( \dot{x} \left( \frac{\pi}{2w} \right) = 0. \)

Sol: \( x(0 = 0 \Rightarrow 0 = c_1 \cos(0) + c_2 \sin(0) = c_1.1 \Rightarrow c_1 = 0. \)
\[ \dot{x}(t) = -c_1w \sin(wt) + c_2w \cos(wt) \]
\[ \dot{x} \left( \frac{\pi}{2w} \right) = 0 \Rightarrow 0 = 0 + c_2w \cos \left( \frac{\pi}{2} \right) = c_2w(0) \]
Since \( w \neq 0 \), therefore \( 0 = 0.c_2 \) which is true for any real value of \( c_2. \)

Thus the required solutions are \( x(t) = c_2 \sin(wt) \) with \( c_2 \) arbitrary.

(b) (3-points) Given \( y_1 = e^x \) and \( y_2 = e^x \tan x \) are two solutions of the differential equation
\[ y'' - 2(1 + \tan x)y' + (1 + 2 \tan x)y = 0. \]
Determine whether or not the set \( \{ y_1, y_2 \} \) form a fundamental set of solutions on the interval \( \left( 0, \frac{\pi}{2} \right) \).

**Sol:** For \( y_1 = e^x, \ y'_1 = e^x \) and for \( y_2 = e^x \tan x, \ y'_2 = e^x \tan x + e^x \sec^2 x \)

Now \( W = \begin{vmatrix} e^x & e^x \tan x \\ e^x & e^x \tan x + e^x \sec^2 x \end{vmatrix} = \frac{1}{\cos^2 x} e^{2x} \neq 0 \) for all \( x \in \left( 0, \frac{\pi}{2} \right) \).

Hence \( \{ y_1, y_2 \} \) forms a fundamental set of solutions.

**Q.3: (a) (3-points)** Find a suitable substitution that transforms the differential equation

\[ xy' + \left( 4x^2 + y^2 \right) dx = 0 \]

to a separable equation. **Find the new equation, but do not find its solution.**

**Sol:** Given DE is a homogeneous equation of degree 2. Let \( y = ux \), then \( dy = xdu + udx \)

Substitute in DE \( \Rightarrow u x^2 ( xdu + udx ) + (4x^2 + u^2 x^2 ) dx = 0 \)

\( \Rightarrow x^2 \left[ u x du + (2u^2 + 4) dx \right] = 0 \ \Rightarrow u x du = -(2u^2 + 4) dx \)

\( \Rightarrow \left( \frac{u}{2u^2 + 4} \right) du = -\frac{1}{x} dx \) which is separable.

**Q.4: (7-points)** Solve the initial value problem \( x \ln (x) \frac{dy}{dx} + \cos^2 (y) = 1, \ y(e) = \frac{\pi}{4} \).

**Sol:** \( x \ln (x) \frac{dy}{dx} + \cos^2 (y) = 1 \)

\( x \ln (x) \frac{dy}{dx} = 1 - \cos^2 (y) \)
\[ x \ln (x) \frac{dy}{dx} = \sin^2 (y) \]
\[ \frac{1}{\sin^2 (y)} dy = \frac{1}{x \ln (x)} dx \]
\[ \csc^2 (y) dy = \frac{1}{x \ln (x)} dx \]
\[ - \cot (y) = \ln (\ln |x|) + C \]
\[ y (e) = \frac{\pi}{4} \Rightarrow - \cot \left( \frac{\pi}{4} \right) = \ln \ln (e) + C \Rightarrow -1 = \ln (1) + C \Rightarrow C = -1. \]
\[ \cot (y) + \ln (\ln |x|) = 1. \]

Q.5: **(6-points)** Find the general solution of the differential equation

\[ ydx = (x + y \ln y) dy. \]

Sol: Given DE is linear in \( x \) and \( \frac{dx}{dy} \) because it can be written as \( \frac{dx}{dy} - \frac{1}{y} x = \ln y. \)

\[ IF = e^{\int -\frac{1}{y} dy} = e^{-\ln y} = \frac{1}{y}. \]

\[ \frac{d}{dy} \left( \frac{x}{y} \right) = \frac{\ln y}{y} \Rightarrow \frac{x}{y} = \int \frac{\ln y}{y} dy = \frac{(\ln y)^2}{2} + C \]

\[ x = \frac{y (\ln y)^2}{2} + Cy. \]

Q.6: **(7-points)** Solve the differential equation \((3x + 4y^2) dx + 4xy dy = 0\) by transforming it into an exact equation.

Sol: \( M (x, y) = 3x + 4y^2 \) and \( N (x, y) = 4xy \)

\[ M_y = 8y \text{ and } N_x = 4y \]

Since \( M_y \neq N_x \), the given DE is not exact. So we need to find an integrating factor.

\[ \frac{M_y - N_x}{N} = \frac{8y - 4y}{4xy} = \frac{4y}{4xy} = \frac{1}{x}, \text{ function of } x \text{ alone.} \]

\[ \mu (x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int \frac{1}{x} dx} = e^{\ln (x)} = x. \]

Multiply given DE by \( x \) to make it exact.
\[(3x^2 + 4xy^2) \, dx + 2x^2y \, dy = 0.\]

\[
\mu(x) \, N(x, y) = \frac{\partial f}{\partial y} = 4x^2y
\]

\[f(x, y) = \int 4x^2y \, dy + h(x)\]

\[f(x, y) = 2x^2y^2 + h(x)\]

\[\frac{\partial f}{\partial x} = 4xy^2 + h'(x) = \mu(x)M(x, y) = 3x^2 + 4xy^2\]

\[h'(x) = 3x^2 \implies h(x) = x^3 + C\]

Thus \[f(x, y) = 2x^2y^2 + x^3 + C = 0\]

OR \[2x^2y^2 + x^3 = C.\]

**Q.7: (6-points)** A glass of water initially at 50°F is placed in a freezer. The freezer is kept at the constant temperature 30°F. After one hour the temperature of the water in glass is 40°F. Find the exact time needed for the temperature of the water to reach 32°F after it is placed in the freezer.

**Sol:** Solve the IVP \[\frac{dT}{dt} = k(T - T_m), \quad T(0) = 50, \quad \text{with} \quad T_m = 30°F.\]

\[T(t) = T_m + ce^{kt} = 30 + ce^{kt}.\]

Using \(T(0) = 50\) we get \(c = 20\). So \(T(t) = 30 + 20e^{kt}\)

Using \(T(1) = 40 \implies 40 = 30 + 20e^k \implies k = -\ln 2 = -\ln \left(\frac{1}{2}\right).\)

\[T(t) = 30 + 20e^{t\ln\left(\frac{1}{2}\right)} = 30 + 20e^{\ln\left(\frac{1}{2}\right)}t = 30 + 20\left(\frac{1}{2}\right)^t.\]

Now \(T(t) = 32 \implies 32 = 30 + 20\left(\frac{1}{2}\right)^t \implies \left(\frac{1}{2}\right)^t = \frac{1}{10} \implies t = \frac{\ln\left(\frac{1}{10}\right)}{\ln\left(\frac{1}{2}\right)} = \frac{\ln 10}{\ln 2}.\]