King Fahd University of Petroleum and Minerals  
Department of Mathematics and Statistics  
Math 460  
Final Exam, Semester II, 2008-2009

Duration: 120 minutes

Name:__________________________________________
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Section:________________________________________

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1. Let \( A = \begin{bmatrix} -\frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & -\frac{\pi}{2} \end{bmatrix} \)

(a) Find the spectral decomposition for \( A \).
(b) Compute \( \cos(A) \).
2. If 1 and \( \frac{9}{10} \) are eigenvalues of a matrix \( B \), where
\[
B = \begin{bmatrix}
\frac{7}{5} & \frac{1}{5} \\
-1 & \frac{1}{2}
\end{bmatrix},
\]
Find \( \lim_{n \to \infty} B^n \).

3. Prove that \( \det(e^A) = e^{\text{Trace}(A)} \).
4. For all $x \in \mathbb{C}^n$, prove that
\[ \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2. \]

5. For $T = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$. Show that $\sigma(T) = \sigma(A) \cup \sigma(C)$, for square $A$ and $C$. 
6. Find the Gerschgorin circles for the eigenvalues of the following matrix

\[ A = \begin{bmatrix} 12i & 1 & 9 & -4 \\ 1 & -6 & 2 + i & -1 \\ 4 & 1 & -1 & 4i \\ 1 - 3i & -9 & 1 & 4 - 7i \end{bmatrix} \]

7. The QR factorization of a matrix \( A \) is given by

\[ Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{3} & \sqrt{3} & -\sqrt{3} \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{6} \end{bmatrix}. \]

Find \( \|A\|_\infty \).
8. Consider two species that coexist in the same environment but compete the same resources. Suppose that the population of each species increases proportionally to the number of its own kind but decreases proportionally to the number in the competing species- say that the population of each species increases at a rate equal to twice its existing number but decreases at a rate equal to the number in the other population. Suppose initially 100 of species I and 200 of species II.

(a) Determine the number of each species at all future times.
(b) Determine which species is destined to become extinct and compute the time to extinction.
9. Consider the following matrix

\[ A = \begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 2
\end{bmatrix} . \]

Find the following

(a) The SVD for \( A \).
(b) \( \text{Rank}(A) \).
(c) Basis for \( R(A) \).
(d) Basis for \( N(A) \).
(e) Use part (a) to calculate \( A^\dagger \).
(f) Compute \( \|A\|_F \) and \( \|A\|_2 \).
(g) The condtional number for \( A \).
10. Prove that if $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ are the nonzero singular values of a rank $r$ matrix $A$, and if $\|E\|_2 < \sigma_r$, then $\text{rank}(A + E) \geq \text{rank}(A)$.
11. Let \( A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix} \) and \( b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). The singular value decomposition for \( A \) is given by

\[
U = \begin{bmatrix} -0.4 & 0.58 & -0.71 \\ -0.82 & -0.56 & 0 \\ -0.4 & 0.58 & 0.71 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5.54 & 0 & 0 \\ 0 & 0.57 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} -0.44 & 0.07 & -0.89 \\ -0.88 & 0.13 & 0.45 \\ -0.15 & -0.99 & 0 \end{bmatrix}
\]

(a) Compute the orthogonal projectors onto each of the four fundamental subspaces associated with \( A \).

(b) Find the point in \( N(A)^\perp \) that is closest to \( b \).
12. Consider the following matrix

\[
A = \begin{bmatrix}
3 & 0 & 1 \\
-4 & 1 & -2 \\
-4 & 0 & -1
\end{bmatrix}
\]

(a) Calculate the algebraic and geometric multiplicity for each eigenvalue of the matrix A.

(b) Find the Jordan form for A.