Problem 1. Consider the differential equation \( z'' + 4z' + 3z = 0 \).
Let \((z, z') = (x, y)\) and set \( Y = \text{col}(x, y)\).
- Write the differential equation as \( Y' = AY(x) \).
- Find the matrix \( \exp(tA) \) and the general solution of \((x)\).
- Find \( M > 0 \) and \( \alpha < 0 \) such that \( \|e^{tA}\| \leq Me^{-\alpha t} \).
- Deduce that \( \lim_{t \to +\infty} |z(t)| = 0 \).

Problem 2. Assume that the functions \( p, q \) are continuous on \([a, b]\) and let \( t_0 \in [a, b] \). Find the solution \( y \) of the differential equation \( y'(t) = p(t)y(t) + q(t) \), that satisfies \( y(t_0) = y_0 \).
- Application. Solve \( y'(t) = -t^{-1}y(t) - t^{-2}, y(1) = 1 \).
- Suppose, further, that \( p \) and \( q \) are periodic of period \( T \).
- Show that the equation \( y' = py + q \), has a periodic solution if and only if \( \exp\int_0^T p(s)\, ds \neq 1 \).
- Find the periodic solution of \( y'(t) = ty(t) + \sin t \).
Problem 3. Consider the periodic differential equation

$$X' = A(t) X$$

where $X = \text{col}(x, y)$ and $A(t) = \begin{pmatrix} 1 & 1 \\ 0 & h(t) \end{pmatrix}$, with

$$h(t) = \frac{\cos t + \sin t}{2 + \sin t - \cos t}$$

1. Verify that $A(t+2\pi) = A(t)$ for every $t$.
2. Prove that $x(t) = be^{t} - a(sint + 2)$, $y(t) = a(2 + sint - \cos t)$, where $a, b$ are arbitrary constants.
3. Find a fundamental matrix solution of (1). Call it $\Phi(t)$. (hint: choose carefully $a$ and $b$.)
4. Find a nonsingular matrix $C$ such that $\Phi(t+2\pi) = \Phi(t)C$.
5. Find the characteristic multipliers of (1).

Problem 4. Let $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 1 & 1 & 1 \end{pmatrix}$

Show that

$$e^{tA} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{e^{\lambda t}}{\lambda} & 1 & 0 \\ \frac{e^{\lambda t}}{2} & \frac{\lambda t}{2} & 1 \end{pmatrix} e^{\lambda t}$$

6. Generalize to an nxn matrix.