MATH 574

Assignment 2 Due Date: 8th of June, 2009

Consider the following parabolic problem:

$$\begin{cases} u_t - u_{xx} + u = 2t(t+1)\sin(x) + 5\sin(2x) \quad \text{for } 0 < x < \pi \text{ and } 0 < t < 1 \\ u(0,t) = u(\pi,t) = 0 \quad \text{for } 0 < t < 1 \\ u(x,0) = \sin(2x) \quad \text{for } 0 < x < \pi. \end{cases}$$

Here u is a function of x and t, that is $u = u(x,t)$.

a) Verify that $u = t^2\sin(x) + \sin(2x)$ is a solution of the given parabolic problem.

b) Define the Crank-Nicolson scheme in time combined with the piecewise linear FEM in space for solving the above problem using uniform partitions in both time and space with $\mathbf{M}$ and $\mathbf{N}$ subintervals, respectively. Assume $h = 1/\mathbf{M}$ and $k = 1/\mathbf{N}$.

c) Write the proposed scheme (in part (b)) in a matrix form.

d) Choose $\mathbf{M} = 30$ and $\mathbf{N} = 10$. Compute the finite element solution at the nodes of the time and spatial partitions. Use the obtained datum to plot the approximate solution. Plot the graph of the exact solution. (Do not plot the approximate and the exact solutions on the same figure.)

e) Say $u_{\mathbf{h}}$ is your approximate solution. Compute the error

$$\max_{0 \le i \le \mathbf{N}} \|u(t_i) - u_{\mathbf{h}}(t_i)\| \quad \text{for } \mathbf{M} = 30, 60, 120, 240 \text{ and } \mathbf{N} = \mathbf{M}/3.$$

Find also the order of convergence and justify whether the obtained numerical results match with the theoretical ones.

f) Discuss in details how did you compute the error, that is, did you use any quadrature rule?, or did you compute the numerical solution from the proposed scheme directly?, etc.