Answer the following questions.

**Question 1:** Find the absolute maximum and absolute minimum of the function 
\[ f(x) = x - \sqrt{x}, \quad x \in [0,4] . \]

Solution:

\[
f'(x) = 1 - \frac{1}{2\sqrt{x}} = \frac{2\sqrt{x} - 1}{2\sqrt{x}}
\]

To find the critical numbers:

\[
f'(x) = 0 \Rightarrow 2\sqrt{x} - 1 = 0 \Rightarrow x = \frac{1}{4}
\]

Now,

\[
f(0) = 0, \quad f(4) = 4 - 2 = 2, \quad \text{and} \quad f\left(\frac{1}{4}\right) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}
\]

Hence, the maximum value of \( f \) is 2 and the minimum value is \(-\frac{1}{4}\).

**Question 2:** If \( f(1) = 10 \) and \( f'(x) \geq 2 \) for \( 1 \leq x \leq 4 \), how small can \( f(4) \) possibly be?

Solution: Since \( f \) is differentiable on \( I = [1,4] \), then it is continuous on \( I \). Therefore, by the mean value theorem there exists \( c \in [1,4] \) such that

\[
f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{f(4) - 10}{3} \Rightarrow f(4) = 3f'(c) + 10 \geq 3 \cdot 2 + 10 = 16
\]

This shows that \( f(4) \) can not be smaller than 16.
Solution

**Question 3:** Sketch the graph of a function $f$ that satisfies all of the following conditions: $f'(x) > 0$ for all $x \neq 1$, vertical asymptote $x = 1$, $f''(x) > 0$ if $x < 1$ or $x > 3$ and $f''(x) < 0$ if $1 < x < 3$.

![Graph of a function](image)

**Question 4:** Find the inflection points of $f(x) = (x^2 - 1)^3$.

Solution: $f'(x) = 6x(x^2 - 1)^2$ and $f''(x) = 6(x^2 - 1)^2 + 24x^2(x^2 - 1) = 6(x^2 - 1)(5x^2 - 1)$.

To find the inflection point: set $f''(x) = 0$. Therefore

$$6(x^2 - 1)(5x^2 - 1) = 0 \Rightarrow x = \pm 1, \pm \frac{\sqrt{5}}{5}$$

Now, we discuss the sign of $f''$.

<table>
<thead>
<tr>
<th>Interval</th>
<th>$f''$</th>
<th>Concavity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty,-1)$</td>
<td>+</td>
<td>Upward</td>
</tr>
<tr>
<td>$(-1,-\frac{\sqrt{5}}{5})$</td>
<td>-</td>
<td>Downward</td>
</tr>
<tr>
<td>$(-\frac{\sqrt{5}}{5},\frac{\sqrt{5}}{5})$</td>
<td>+</td>
<td>Upward</td>
</tr>
<tr>
<td>$(\frac{\sqrt{5}}{5},1)$</td>
<td>-</td>
<td>Downward</td>
</tr>
<tr>
<td>$(1,\infty)$</td>
<td>+</td>
<td>Upward</td>
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</tbody>
</table>

Hence, the inflection points are $(\pm 1,0), \left(\pm \frac{\sqrt{5}}{5}, -\frac{64}{125}\right)$. 