EXAM I CODE 003

$\boxed{Q1}$ The slope of the tangent line at the point $(0, 1)$ to the curve $(y-1)\ln(e+x) = xe^y$ is given by

$$\frac{d}{dx}\left[(y-1)\ln(e+x)\right] = \frac{d}{dx}(xe^y)$$

$$\Rightarrow \frac{dy}{dx}\ln(e+x) + \frac{y-1}{e+x} = e^y + x\frac{dy}{dx}e^y$$

When $x=0$ & $y=1$, we obtain:

$$\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=1}} \ln(e+0) + \frac{1-1}{e+0} = e^1 + 0.\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=1}} e^1$$

Thus $\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=1}} = e$.

a) $\frac{1}{e}$

b) -1

c) $\frac{e+1}{e-1}$

$\boxed{\text{d) } e}$

e) $\frac{e}{1-e}$

$\boxed{Q2}$ The slope of the tangent line at $x = 2$ to the curve $y = \sqrt{\log_2 x}$ is

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{d}{dx}(\log_2 x)\cdot(\log_2 x)^{\frac{1}{2}-1}$$

$$= \frac{1}{2}\cdot\frac{1}{x\ln 2}\cdot\frac{1}{\sqrt{\log_2 x}}$$

Since $\log_2 2 = \frac{\ln 2}{\ln 2} = 1$, $\left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{4\ln 2}$

a) $\frac{1}{\ln 2}$

b) $\frac{1}{2\sqrt{\log_2 2}\ln 2}$

c) $\frac{\ln 2}{2}$

d) $\frac{1}{2\sqrt{\log_2 2}}$

$\boxed{\text{e) } \frac{1}{4\ln 2}}$

$\boxed{Q3}$ $\lim_{x\to -1} \frac{x^3-x}{3x^2+2x-1}$ $= \lim_{x\to -1} \frac{x(x^2-1)}{3(x+1)(x-\frac{1}{3})} = \lim_{x\to -1} \frac{x(x-1)(x+1)}{(x+1)(3x-1)}$

$= \lim_{x\to -1} \frac{x(x-1)}{3x-1} = -\frac{1}{2}$

a) Does not exist

b) $-\infty$

c) $+\infty$

d) $-\frac{1}{2}$

e) 0

$\boxed{Q4}$ If $y = \ln(u+2)$ and $u = x^2 - 2x - 1$, then

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u+2} \cdot (2x-2)$

When $x = 0$, $u(0) = -1$. Therefore

$\left.\frac{dy}{dx}\right|_{x=0} = \frac{1}{-1+2} \cdot (-2) = -2$

a) $1 \le \frac{dy}{dx}\,|_{x=0} < 3$

b) $5 \le \frac{dy}{dx}\,|_{x=0}$

c) $-5 \le \frac{dy}{dx}\,|_{x=0} < 1$

d) $3 \le \frac{dy}{dx}\,|_{x=0} < 5$

e) $\frac{dy}{dx}\,|_{x=0} < -5$

$\boxed{Q5}$ $\lim_{x\to-\infty} \frac{1-2x+x^3-6x^5}{x^3-x^2+5}$ $= \lim_{x\to-\infty} \frac{-6x^5}{x^3} = \lim_{x\to-\infty} -6x^2 = -\infty$

a) -6

b) $-\infty$

c) 1

d) 0

e) $+\infty$

$\boxed{Q6}$ If $f(x) = \sqrt[3]{\frac{1-2x}{1-3x}}$, then

a) $f'(x) = \frac{1}{3(1-3x)^{\frac{5}{2}}(1-2x)^{-\frac{1}{2}}}$

b) $f'(x) = \frac{(1-2x)^{-\frac{2}{3}}}{3(1-3x)^{\frac{4}{3}}}$

c) $f'(x) = \frac{1}{(1-3x)^{\frac{4}{3}}(1-2x)^{\frac{1}{3}}}$

d) $f'(x) = \frac{(1-2x)^{-\frac{1}{3}}}{3(1-3x)^{-\frac{2}{3}}}$

e) $f'(x) = \frac{(1-2x)^{-\frac{2}{3}}}{3(1-3x)^{-\frac{2}{3}}}$

$$f'(x) = \frac{1}{3}\cdot\frac{d}{dx}\left(\frac{1-2x}{1-3x}\right)\cdot\left(\frac{1-2x}{1-3x}\right)^{\frac{1}{3}-1}$$

$$= \frac{1}{3}\ \frac{-2(1-3x)+3(1-2x)}{(1-3x)^2}\left(\frac{1-2x}{1-3x}\right)^{-\frac{2}{3}}$$

$$= \frac{1}{3}\ \frac{1}{(1-3x)^2}\cdot\frac{(1-2x)^{-\frac{2}{3}}}{(1-3x)^{2/3}}$$

$$= \frac{(1-2x)^{-\frac{2}{3}}}{3(1-3x)^{2-\frac{2}{3}}}$$

$$= \frac{(1-2x)^{-\frac{2}{3}}}{3(1-3x)^{\frac{4}{3}}}$$

$\boxed{Q7}$ If $y = (\frac{1}{2x-1})^{x^2}$, then

a) $-\frac{1}{4} \le \frac{dy}{dx}|_{x=1} < 0$

$\boxed{\text{b) } -\frac{5}{2} \le \frac{dy}{dx}|_{x=1} < -1}$

c) $-1 \le \frac{dy}{dx}|_{x=1} < -\frac{1}{4}$

d) $0 \le \frac{dy}{dx}|_{x=1} < \frac{1}{4}$

e) $\frac{1}{4} \le \frac{dy}{dx}|_{x=1} < 2$

$$y = e^{x^2 \ln\left(\frac{1}{2x-1}\right)} = e^{-x^2\ln(2x-1)}$$

$$\frac{dy}{dx} = e^{-x^2\ln(2x-1)} \cdot \frac{d}{dx}\left(-x^2\ln(2x-1)\right)$$

$$= \left[-2x\ln(2x-1) - \frac{2x^2}{2x-1}\right] e^{-x^2\ln(2x-1)}$$

So.

$$\left.\frac{dy}{dx}\right|_{x=1} = -2$$

$\boxed{Q8}$ An equation of the tangent line to the curve

$$y = e^{\frac{2}{x} - \sqrt{x}}$$

at $x = 1$ is given by

a) $y = -\frac{5e}{2}x - \frac{3e}{2}$

b) $y = -\frac{5e}{2}(x+1) + e$

c) $y = \frac{e}{2}(3 - 5x)$

d) $y = -\frac{5e}{2}(x-1)$

$\boxed{\text{e) } y = \frac{e}{2}(7 - 5x)}$

•) $\frac{dy}{dx} = \left(-\frac{2}{x^2} - \frac{1}{2\sqrt{x}}\right) e^{\frac{2}{x} - \sqrt{x}}$

•) The slope of the tangent line at $x=1$ is:

$$\left.\frac{dy}{dx}\right|_{x=1} = \left(-2 - \frac{1}{2}\right) e^{2-1} = -\frac{5}{2}e$$

•) An equation of the tangent line is:

$$y = -\frac{5}{2}e(x-1) + e$$

$$= \frac{e}{2}\left[-5(x-1) + 2\right] = \frac{e}{2}(7 - 5x)$$

$\boxed{Q9}$ The points of discontinuity of $f(x) = \dfrac{9-x^2}{x^3-4x^2+3x}$ are

a) 0 and 1

b) -3 and 1

$\boxed{\text{c) } 0, 1 \text{ and } 3}$

d) -3, -1 and 0

e) -1 and 0

$f(x)$ is well defined iff $x^3-4x^2+3x \neq 0$
iff $x(x^2-4x+3) \neq 0$
iff $x(x-1)(x-3) \neq 0$
iff $x \neq \{0, 1, 3\}$.

Since f is a rational function, it is continuous only on its domain of definition which is $\mathbb{R} \setminus \{0, 1, 3\}$.
Therefore f is discontinuous at 0, 1 and 3

$\boxed{Q10}$ If $1 - e^x = -y + \ln(xy)$, then $\frac{dy}{dx}$ is equal to

a) $\dfrac{y(1-xe^x)}{x(y+1)}$

b) $\dfrac{-y(1+e^x)}{x+y}$

c) $\dfrac{y+xe^x}{x+\ln(xy)}$

$\boxed{\text{d) } \dfrac{y(1+xe^x)}{x(y-1)}}$

e) $\dfrac{x+y\ln(xy)}{y(x-1)}$

$$\frac{d}{dx}(1-e^x) = \frac{d}{dx}\left[-y+\ln(xy)\right]$$

$$-e^x = -\frac{dy}{dx} + \frac{y + x\frac{dy}{dx}}{xy}$$

$$\Rightarrow -xye^x = (-xy+x)\frac{dy}{dx} + y$$

$$\Rightarrow \frac{-y(1+xe^x)}{x(-y+1)} = \frac{dy}{dx}$$

$$\Rightarrow \frac{y(1+xe^x)}{x(y-1)} = \frac{dy}{dx}$$

$\boxed{Q11}$ If $y = (\ln 5)x^5 \log_5 x$, then

a) $\frac{dy}{dx}\,|_{x=e} = 5e^4 + \frac{e^4}{\ln 5}$

$\boxed{\text{b) } \frac{dy}{dx}\,|_{x=e} = 6e^4}$

c) $\frac{dy}{dx}\,|_{x=e} = 5e^4 \log_5 e$

d) $\frac{dy}{dx}\,|_{x=e} = 5e^4$

e) $\frac{dy}{dx}\,|_{x=e} = 5e^4 \log_5 e + \frac{e^5}{\ln 5}$

$$\frac{dy}{dx} = (\ln 5)\left[5x^4 \cdot \log_5 x + \frac{x^5}{x \ln 5}\right]$$
$$= 5x^4 \ln x + x^4$$

Hence: $\left.\frac{dy}{dx}\right|_{x=e} = 5e^4 + e^4 = 6e^4$

$\boxed{Q12}$ An equation of the tangent line at $x = 1$ to the curve $y = \frac{1-x^2}{x^2+1}$ is given by

a) $y = 4(x-1)$

$\boxed{\text{b) } y = 1 - x}$

c) $y = x - 1$

d) $y = 2x - 1$

e) $y = 2(1-x)$

•) $\frac{dy}{dx} = \frac{-2x(x^2+1) - 2x(1-x^2)}{(x^2+1)^2} = \frac{-4x}{(x^2+1)^2}$

•) The slope of the tangent line at $x = 1$ is:

$$\left.\frac{dy}{dx}\right|_{x=1} = -\frac{4}{2^2} = -1$$

•) An equation of the tangent line is:

$$y = -1(x-1) + 0 = 1 - x$$

$\boxed{Q13}$ The function $g(x) = \begin{cases} \frac{1-5x}{x(x-1)}, & \text{if } x \leq -1 \\ \frac{x^3+1}{x+1}, & \text{if } x > -1 \end{cases}$

a) is discontinuous at both 0 and 1

b) is discontinuous at -1, 0 and 1

c) is continuous everywhere

d) is discontinuous only at -1

e) is discontinuous at both -1 and 1

•) $\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} \frac{1-5x}{x(x-1)} = 3$

$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} \frac{x^3+1}{x+1} =$

$\lim_{x \to -1^+} \frac{(x+1)(x^2-x+1)}{x+1} = \lim_{x \to -1^+} x^2-x+1$

$= 3$. Hence $\lim_{x \to -1} g(x) = 3$

Since $g(-1) = 3$, g is continuous at -1

•) On $(-\infty, -1)$, g is a rational function which is well defined on $(-\infty,-1)$, so is continuous on $(-\infty,-1)$

•) On $(-1, +\infty)$, g is a rational function well defined, so is continuous on $(-1, +\infty)$.

$\boxed{Q14}$ If $g(t) = 2\ln\sqrt{\frac{t^2}{1+t}}$, then

a) $-1 < g'(1) \leq 0$

b) $2 < g'(1)$

c) $-2 < g'(1) \leq -1$

d) $0 < g'(1) \leq 2$

e) $g'(1) \leq -2$

$g(t) = 2\ln\left(\frac{t^2}{1+t}\right)^{\frac{1}{2}} = \ln\left(\frac{t^2}{1+t}\right)$

$= \ln t^2 - \ln(1+t)$

$= 2\ln t - \ln(1+t)$.

So $g'(t) = \frac{2}{t} - \frac{1}{1+t}$

Thus $g'(1) = 2 - \frac{1}{2} = \frac{3}{2}$.

$\boxed{Q15}$ If $f(x) = 2^{\ln x}$, then

a) $f^{(2)}(x) = \frac{\ln 2}{x}\left[f'(x) - \frac{f(x)}{x}\right]$

b) $f^{(2)}(x) = \left(\frac{\ln 2}{x}\right)^2 f'(x)$

c) $f^{(2)}(x) = (\ln 2)\left[\frac{f'(x)}{x} + \frac{f(x)}{x^2}\right]$

d) $f^{(2)}(x) = \frac{\ln 2}{x} f(x)$

e) $f^{(2)}(x) = \left(\frac{\ln 2}{x}\right)^2 [f'(x) - f(x)]$

$f'(x) = \frac{\ln 2}{x} f(x)$. So

$$f^{(2)}(x) = \ln 2 . \left[-\frac{1}{x^2} f(x) + \frac{f'(x)}{x}\right] = \frac{\ln 2}{x}\left[f'(x) - \frac{f(x)}{x}\right]$$

$\boxed{Q16}$ If $xy + e^x = y^2$, then

a) $3 \leq \frac{d^2y}{dx^2} \mid x = 0, y = 1$

b) $-\frac{1}{2} \leq \frac{d^2y}{dx^2} \mid x = 0, y = 1 < 0$

c) $0 \leq \frac{d^2y}{dx^2} \mid x = 0, y = 1 < \frac{3}{2}$

d) $\frac{3}{2} \leq \frac{d^2y}{dx^2} \mid x = 0, y = 1 < 3$

e) $\frac{d^2y}{dx^2} \mid x = 0, y = 1 < -\frac{1}{2}$

$$\frac{d}{dx}\left(xy + e^x\right) = \frac{d}{dx} y^2$$

$$\Rightarrow y + x\frac{dy}{dx} + e^x = 2y\frac{dy}{dx} \quad (*)$$

Thus:

$$1 + 1 = 2\left.\frac{dy}{dx}\right|_{x=0, y=1}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0, y=1} = 1.$$

Now differentiate both sides of $(*)$ with respect to x:

$$\frac{d}{dx}\left(y + x\frac{dy}{dx} + e^x\right) = \frac{d}{dx}\left(2y\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} + e^x = 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}.$$

Taking $x = 0$ & $y = 1$, implies.

$$2\left.\frac{dy}{dx}\right|_{x=0, y=1} + 1 = 2\left(\left.\frac{dy}{dx}\right|_{x=0, y=1}\right)^2 + 2\left.\frac{d^2y}{dx^2}\right|_{x=0, y=1}$$

Hence: $\left.\frac{d^2y}{dx^2}\right|_{x=0, y=1} = \frac{1}{2}$

$\boxed{Q17}$ $\lim_{x \to -\frac{1}{2}^-} \frac{x-1}{2x^2-x-1} = \lim_{x \to -\frac{1}{2}^-} \frac{(x-1)}{2(x-1)(x+\frac{1}{2})} = \lim_{x \to -\frac{1}{2}^-} \frac{1}{2(x+\frac{1}{2})}$

$= -\infty$, Since $\lim_{x \to -\frac{1}{2}^-} x + \frac{1}{2} = 0^-$

a) $-\frac{1}{4}$

b) $+\infty$

$\boxed{\text{c) } -\infty}$

d) 0

e) $\frac{1}{2}$

$\boxed{Q18}$ The derivative of $h(x) = (1-3x)^{-\frac{2}{3}} + \frac{5}{\sqrt[5]{x^2}}$ is

$h(x) = (1-3x)^{-\frac{2}{3}} + 5x^{-\frac{2}{5}}$.

So

$h'(x) = -\frac{2}{3}(-3)(1-3x)^{-\frac{2}{3}-1} + 5\left(-\frac{2}{5}\right)x^{-\frac{2}{5}-1}$

$= 2(1-3x)^{-\frac{5}{3}} - 2x^{-7/5}$

$= 2\left[\frac{1}{(1-3x)^{\frac{5}{3}}} - \frac{1}{x^{7/5}}\right]$

a) $2(1-3x)^{-\frac{5}{3}} - 5x^{-\frac{7}{5}}$

b) $2(1-3x)^{-\frac{5}{3}} + \frac{2}{x^{\frac{7}{5}}}$

c) $\frac{1}{(1-3x)^{\frac{5}{3}}} - \frac{1}{x^{\frac{7}{5}}}$

d) $-\frac{2}{3}(1-3x)^{-\frac{5}{3}} + 2x^{-\frac{7}{5}}$

$\boxed{\text{e) } 2\left[\frac{1}{(1-3x)^{\frac{5}{3}}} - \frac{1}{x^{\frac{7}{5}}}\right]}$

$\boxed{Q19}$ If $f(x) = e^{(x^2+1)}3^{(x+1)\ln x}$, then $f'(1)$ is equal to

$\boxed{\text{a) } 2e^2(1+\ln 3)}$

b) $2e^2(1+2\ln 3)$

c) $2e^2+\ln 3$

d) $e^2(2+\ln 3)$

e) $e^2(1+2\ln 3)$

$f'(x) = 2x e^{(x^2+1)} \, 3^{(x+1)\ln x} + e^{(x^2+1)}(\ln 3)\left[\ln x + \frac{x+1}{x}\right]$

Thus:

$f'(1) = 2e^2 + 2e^2 \ln 3$
$= 2e^2(1+\ln 3)$

$\boxed{Q20}$ If $y = x^{\ln x}$, then $y''(e)$ is equal to

a) $\frac{2}{e}$

b) $2e + \frac{4}{e^2}$

c) $\frac{2}{e^2} + \frac{4}{e}$

$\boxed{\text{d) } \frac{4}{e}}$

e) $\frac{2}{e} + \frac{4}{e^2}$

$y = e^{(\ln x)^2}$

So $y' = \frac{2}{x}(\ln x)\, e^{(\ln x)^2}$

and

$y'' = 2\left[\left(-\frac{\ln x}{x^2} + \frac{1}{x^2}\right) e^{(\ln x)^2} + 2\frac{(\ln x)^2}{x^2} e^{(\ln x)^2}\right]$

Therefore

$y''(e) = \frac{4}{e}$