Implicit Differentiation:

Exercise 40 (page 214)

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0. \] Hence \( y' = -\frac{b^2 x}{a^2 y} \). Thus the equation of the tangent line at \((x_0, y_0)\) is

\[ y - y_0 = -\frac{b^2 x_0}{a^2 y_0} (x - x_0). \]

If we multiply both sides by \( \frac{y_0}{b^2} \), we get

\[ \frac{y_0 y}{b^2} - \frac{y_0}{b^2} = -\frac{x_0}{a^2} + \frac{x_0^2}{a^2}. \]

Since, in addition, \( \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1 \) we have

\[ \frac{y_0 y}{b^2} + \frac{x_0 x}{a^2} = 1 \] (an equation of the tangent to the ellipse at the point \((x, y_0)\)).

Exercise 41

Let \((x_0, y_0)\) be a point of the hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \). By implicit differentiation with respect to \( x \), we have

\[ \frac{2x}{a^2} - \frac{2yy'}{b^2} = 0. \]

This gives \( y' = \frac{b^2 x}{a^2 y} \). Hence, an equation of the tangent line at \((x_0, y_0)\) is

\[ y - y_0 = \frac{b^2 x_0}{a^2 y_0} (x - x_0). \]

We multiply by \( \frac{y_0}{b^2} \) both sides. This leads to

\[ \frac{y_0 y}{b^2} - \frac{y_0^2}{b^2} = \frac{x_0 x}{a^2} - \frac{x_0^2}{a^2}. \]

But, since \((x_0, y_0)\) is a point of the hyperbola, we have:

\[ \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1. \]

Thus, \( \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1 \) is an equation of the tangent line to the hyperbola at \((x_0, y_0)\).
Exercise 68:
(a) Let \( f(x) = 2x + \cos x \). Then \( f'(x) = 2 - \sin x \). Hence \( f'(x) > 0 \), for each \( x \). Thus \( f \) is increasing, therefore, \( f \) is one-to-one.
(b) Let \( k = f^{-1}(1) \); that is to say \( f(k) = 1 \).
Since \( f(\theta) = 2\theta + \cos \theta = 1 \), we have \( k = f^{-1}(1) = 0 \).

(c) \( (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \).

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} = \frac{1}{2}.
\]

Exercise 67:
(a) \( y = f^{-1}(x) \iff x = f(y) \). Differentiating implicitly with respect to \( x \), we get

\[
1 = f'(y) \frac{dy}{dx}.
\]
Hence \( \frac{dy}{dx} = \frac{1}{f'(y)} \), that is to say \( (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \).
(b) if \( f(4) = 5 \), then \( f^{-1}(5) = 4 \), and thus \( (f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(4)} \).
Exercise 4.3

Consider the curve of equation $\sqrt{x} + \sqrt{y} = \sqrt{c}$ (c is a given positive number).

By implicit differentiation with respect to $x$, we have

$$\frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0.$$ 

Thus, an equation of the tangent line to the curve at a point $(x_0, y_0)$ is

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}} (x - x_0).$$

To find the $y$-intercept of the tangent, it suffices to let $x = 0$ in its equation. This gives

$$y = y_0 - \frac{\sqrt{y_0}}{\sqrt{x_0}} (-x_0) = y_0 + \sqrt{x_0} \sqrt{y_0},$$

Therefore, the $y$-intercept is $y = y_0 + \sqrt{x_0} \sqrt{y_0}$.

Now, to find the $x$-intercept of the tangent, it suffices to let $y = 0$ in its equation. This yields

$$-y = -\frac{\sqrt{y_0}}{\sqrt{x_0}} (x - x_0).$$

So that,

$$x = x_0 + \frac{\sqrt{x_0}}{\sqrt{y_0}} y_0 = x_0 + \sqrt{x_0} \sqrt{y_0}.$$

It follows that the $x$-intercept of the tangent line is $x = x_0 + \sqrt{x_0} \sqrt{y_0}$.

The sum of the intercepts is

$$y_0 + \sqrt{x_0} \sqrt{y_0} + x_0 + \sqrt{x_0} \sqrt{y_0} = x_0 + 2\sqrt{x_0} \sqrt{y_0} + y_0 = \frac{(\sqrt{x_0} + \sqrt{y_0})^2}{\sqrt{c}} = \sqrt{c} = c.$$