Related Rates:

Exercise 19 (page 246):

Let \( b \) be the base of the triangle and \( h \) its altitude at time \( t \).
The area of the triangle at time \( t \) is \( A = \frac{1}{2} bh \).

By differentiating with respect to time \( t \), we get

\[
\frac{dA}{dt} = \frac{1}{2} \left( \frac{db}{dt} h + b \frac{dh}{dt} \right).
\]

\[
\Rightarrow \frac{db}{dt} = \frac{2 \frac{dA}{dt} - b \frac{dh}{dt}}{h}.
\]

If \( h = 10 \) and \( A = 100 \), then \( b = \frac{2 \frac{dA}{dt}}{h} = \frac{2 \times 0}{10} = 0 \).

Thus \( \frac{db}{dt} = \frac{2 \times 2 - 20 \times 1}{10} = -1.6 \text{ cm/min} \)

Exercise 20:

What is known is \( \frac{dy}{dt} = -1 \text{ m/s} \). We have to find \( \frac{dx}{dt} \) when \( x = 8 \). By Pythagorean theorem, we have \( y = x^2 + 1 \).

If we differentiate with respect to \( x \), then we get:

\[
2 \frac{dy}{dt} = 2x \frac{dx}{dt}.
\]

Thus, \( \frac{dx}{dt} = \frac{\frac{dy}{dt}}{x} \).

When \( x = 8 \), we have \( y = \sqrt{64+1} = \sqrt{65} \).

Therefore, \( \frac{dx}{dt} = -\frac{\sqrt{65}}{8} \text{ m/s} \).
Exercise 2.2:

Let $D$ be the distance between the origin and the particle $(x, y)$ (with $y = \sqrt{x}$).

Then $D = \sqrt{(x-x_0)^2 + (y-y_0)^2} = \sqrt{x^2 + (\sqrt{x})^2} = \sqrt{x^2 + x}$.

Hence $\frac{dD}{dt} = \frac{1}{2} (3x+1) \left( \frac{1}{\sqrt{x^2 + x}} \right) \left( \frac{dx}{dt} \right)$.

If $x = 4$ and $\frac{dx}{dt} = 3 \text{ cm/s}$, then $\frac{dD}{dt} = \frac{9}{2} \times 3 \text{ cm/s}$.

Exercise 2.3:

Let $C$ be the rate at which water is pumped into the tank. Since water is leaking out of the tank at a rate of $10^4 \text{ cm}^3/\text{min}$, the rate of change of the volume of water in the tank is

$$\frac{dV}{dt} = (C - 10^4) \text{ cm}^3/\text{min}.$$ 

Let $r$ be the radius of the surface of water at the top and $h$ be the level of water. Then the volume of water is

$$V = \frac{1}{3} \pi r^2 h.$$ 

Since the height of the conical tank is 6 m and its diameter at the top is 4 m, we have the following two triangles:

Thus $\frac{r}{h} = \frac{2}{6} = \frac{1}{3}$. This gives $r = \frac{1}{3} h$.

Consequently, $V = \frac{1}{3} \pi \left(\frac{1}{3} h\right)^2 h = \frac{\pi}{27} h^3$.

Now, by differentiating with respect to time, we get

$$\frac{dV}{dt} = \frac{\pi}{9} h^2 \left( \frac{dh}{dt} \right).$$
If \( h = 200 \text{ cm} \) and \( \frac{dh}{dt} = 20 \text{ cm/min} \), then

\[
C = 10^4 = \frac{\pi}{9} (200^2)(20)
\]

\[
\Rightarrow C = 10^4 + \frac{8 \times 10^5}{9} \pi \text{ cm}^3/\text{min}
\]

**Exercise 2.9:**

![Diagram of triangle](image)

The rate of change of the angle \( \theta \) is \( \frac{d\theta}{dt} = 0.06 \text{ rad/s} \).

The area of the triangle is \( A = \frac{5h}{2} \).

Remark that \( \sin \theta = \frac{h}{4} \), thus \( h = 4 \sin \theta \) and so that

\[
A = \frac{5}{2} (4 \sin \theta) = 10 \sin \theta.
\]

By differentiating with respect to time, we get

\[
\frac{dA}{dt} = 10 (\cos \theta) \frac{d\theta}{dt}
\]

When \( \theta = \frac{\pi}{3} \) and \( \frac{d\theta}{dt} = 0.06 \), we have:

\[
\frac{dA}{dt} = 10 \left( \frac{1}{2} \right) 0.06 = 0.3 \text{ m}^2/\text{s}.
\]