King Fahd University of Petroleum & Minerals  
Department of Mathematics & Statistics  
Math 202 Final Exam  
The First Semester of 2009-2010 (091)  
Time Allowed: 180 Minutes

Name: ___________________________ ID#: ______________________
Section/Instructor: ___________________ Serial #: __________________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

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Q.1: Given the following differential equation

\[ \cos x \, dx + \left(1 + \frac{2}{y}\right) \sin x \, dy = 0. \]

(a) (2-points) Determine whether the differential equation is EXACT or not?

(b) (4-points) Find integrating factor and make the differential equation EXACT.

(c) (6-points) Solve the EXACT differential equation obtained in part (b).
Q.2: (10-points) Solve the differential equation using an appropriate substitution

\[
(1 + x^2) \frac{dy}{dx} = xy(y^2 - 1).
\]
Q.3: (12-points) Use annihilator approach to solve the differential equation

\[ y'' + 2y' + y = \cos^2 x - \sin^2 x. \]
Q.4: (10-points) Use variation of parameters method to find particular solution $y_p$ of the differential equation

$$y'' + 4y = \csc 2x.$$
Q.5: (12-points) Solve the following initial value problem

\[ 2x^2 y'' - 3xy' + 3y = 2x^3, \quad y(1) = 0, \quad y'(1) = 1. \]
Q.6: Consider the differential equation

\[ xy'' - xy' + y = 0. \]

(a) (2-points) Show that \( x = 0 \) is a regular singular point of the differential equation.

(b) (3-points) Write the indicial equation for the differential equation and show that indicial roots \( r_1 \) and \( r_2 \) satisfy \( r_1 - r_2 = 1 \).

(c) (10-points) Find a series solution of the differential equation of the form \( y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1} \).
(d) (8-points) Use the reduction of order formula to show that the differential equation has a second series solution of the form 

\[ y_2 = x \left( -\frac{1}{x} + \ln x + \sum_{n=2}^{\infty} \frac{x^{n-1}}{(n-1)n!} \right) \left( \text{Hint: } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \]
Q.7: Consider the following linear system of differential equations $X' = AX$.

(a) (3-points) Write the characteristic equation corresponding to the coefficient matrix

$$A = \begin{bmatrix} -3 + s & -3 \\ 3 & 3 \end{bmatrix},$$

with $s$ a scalar parameter.

(b) (3-points) Find values of $s$ for which the matrix $A$ has complex eigenvalues.

(c) (8-points) Solve the linear system for $s = 6$. 
Q.8: Consider the following linear system

\[
\begin{align*}
\frac{dx}{dt} &= y + z \\
\frac{dy}{dt} &= 2x + y \\
\frac{dz}{dt} &= -z
\end{align*}
\]

(a) (10-points) Write the system into matrix form $X' = AX$ and find eigenvalues and corresponding eigenvectors of the matrix $A$.

(b) (2-points) Find the general solution of the system.
Q.9: (10-points) Consider the nonhomogeneous system

\[ X' = \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix} X + \begin{bmatrix} -3 \\ 6 \end{bmatrix}, \]

Let \( \Phi(t) = \begin{bmatrix} e^{3t} & -e^{-3t} \\ 5e^{3t} & e^{-3t} \end{bmatrix} \) be the fundamental matrix of the associated homogeneous system. Use variation of parameters method to find particular solution \( X_p \) and write the general solution.
Q.10: Let the matrix $A$ be given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

(a) (2-points) Find the matrix $B$ such that $A = I + B$.

(b) (2-points) Show that $B^3 = O$, (the zero matrix)

(c) (3-points) Using $(1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!}x^2 + \frac{n(n - 1)(n - 2)}{3!}x^3 + ...$
and part (b) show that

$$A^n = I + nB + \frac{n(n - 1)}{2}B^2, \ n = 0, 1, 2, 3, ...$$

(d) (8-points) Compute the matrix exponential $e^{At}$ and write the solution of $X' = AX$. 