Exercise 1. Let $f$ be the complex function defined by $f(z) = \frac{z - i}{z + i}$.

(1) Find the real functions $u(x, y), v(x, y)$ such that 

$$f(x + iy) = u(x, y) + iv(x, y).$$

(2) Find all points at which the Cauchy-Riemann equations are satisfied.

Exercise 2. Determine the set of all complex numbers $z$ such that

$$|z + 1 + 6i| = |z - 3 + i|.$$
Exercise 1. Of course, the domain of \( f \) is \( \mathbb{C} \setminus \{i\} \). Let \( z = x + iy \in \mathbb{C} \setminus \{-i\} \), with \( x = \text{Re}(z) \) and \( y = \text{Im}(z) \). The we have:

\[
    f(z) = \frac{x + i(y - 1)}{x + i(y + 1)}
\]

\[
    = \frac{(x + i(y - 1)) (x - i(y + 1))}{(x + i(y + 1)) (x - i(y + 1))}
\]

\[
    = \frac{x^2 + (y^2 - 1)}{x^2 + (y + 1)^2} + i \left( \frac{-xy - x + xy - x}{x^2 + (y + 1)^2} \right)
\]

\[
    = \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} + i \left( \frac{-2x}{x^2 + (y + 1)^2} \right)
\]

(1) The required real functions are

\[
    u(x, y) = \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} \quad \text{and} \quad v(x, y) = \frac{-2x}{x^2 + (y + 1)^2}
\]

(2) The Cauchy-Riemann equations at the point \((a, b)\) are

\[
    \frac{\partial u}{\partial x}(a, b) = \frac{\partial v}{\partial y}(a, b) \quad \text{and} \quad \frac{\partial u}{\partial y}(a, b) = -\frac{\partial v}{\partial x}(a, b).
\]

We have to compute the first partial derivatives of \( u \) and \( v \).

We have

\[
    \frac{\partial u}{\partial x} = \frac{2x(x^2 + (y + 1)^2) - 2x(x^2 + y^2 - 1)}{(x^2 + (y + 1)^2)^2}
\]

\[
    = \frac{(2x)(2y) + 2x + 2x}{(x^2 + (y + 1)^2)^2}
\]

\[
    = \frac{4x(y + 1)}{(x^2 + (y + 1)^2)^2},
\]

and

\[
    \frac{\partial v}{\partial y} = \frac{-2x - 2(y + 1)}{(x^2 + (y + 1)^2)^2}
\]

\[
    = \frac{4x(y + 1)}{(x^2 + (y + 1)^2)^2}.
\]

It follows that \( \frac{\partial u}{\partial x}(a, b) = \frac{\partial v}{\partial y}(a, b) \), for each \((a, b) \neq (0, -1)\).
Now, we compute the following
\[
\frac{\partial u}{\partial y} = \frac{2y(x^2 + (y + 1)^2) - 2(y + 1)(x^2 + y^2 - 1)}{(x^2 + (y + 1)^2)^2}
\]
\[
= \frac{-2x^2 + 2(y + 1)^2}{(x^2 + (y + 1)^2)^2}
\]
\[
= \frac{2((y + 1)^2 - x^2)}{(x^2 + (y + 1)^2)^2},
\]
and
\[
\frac{\partial v}{\partial x} = \frac{-2(x^2 + (y + 1)^2) + 2x(2x)}{(x^2 + (y + 1)^2)^2}
\]
\[
= \frac{2x^2 - 2(y + 1)^2}{(x^2 + (y + 1)^2)^2}
\]
\[
= \frac{-2((y + 1)^2 - x^2)}{(x^2 + (y + 1)^2)^2}.
\]
Therefore, \(\frac{\partial u}{\partial y}(a, b) = -\frac{\partial v}{\partial x}(a, b)\), for each \((a, b) \neq (0, -1)\).

We conclude that the Cauchy-Riemann equations are satisfied for all \((a, b) \neq (0, -1)\), as expected (since \(f\) is differentiable on \(\mathbb{C} - \{-i\}\)).

**Exercise 2.** Let \(A = (-1, -6)\), \(B = (3, -1)\) and \(M = (x, y)\) be the points of the plane corresponding respectively to the complex numbers \(a = -1 - 6i\), \(b = 3 - i\) and \(z = x + iy\). If we let \(I = A \ast B\) be the midpoint of the segment \([AB]\), then \(z\) satisfies the given equation if and only if \(\vec{IM} \perp \vec{AB}\); that is to say, \(\vec{IM} \cdot \vec{AB} = 0\). Hence, the given equation is equivalent to
\[
4(x - 1) + 5(y + \frac{7}{2}) = 0, \text{ or equivalently } 4x + 5y + \frac{27}{2} = 0.
\]