1. Let $U$ be a vector space and let $f \in U^*$ be a nonzero functional. Show that there is a vector $u \in U$ such that $f(u) = 1$. Discuss the solutions of the equation $f(u) = 1$.

2. Let $U$ be a vector space and let $V$ be a proper subspace of $U$. Pick a vector $u_0 \in U - V$. Show that there is $f_0 \in U^*$ such that $f_0(u_0) = 1$ and $f_0(u) = 0$ for all $u \in V$.

3. Let $U$ be a vector space over a field $F$ and let $A : U \rightarrow U$ be a linear transformation. If $\lambda, \mu \in F$ with $\lambda \neq \mu$ show that $N(A^T - \mu I) \subset N(A - \lambda I)$.

4. Let $U$ be a vector space and let $P : U \rightarrow U$ be a projection. Show that the eigenvalues of $P$ are 0 and 1, $N(P) = W(0)$ and $R(P) = W(1)$. Show further that $U = W(0) \bigoplus W(1)$. Here $W(\lambda)$ is the eigenspace corresponding to $\lambda$.

5. Let $X$ be a nonempty set. A function $d : X \times X \rightarrow \mathbb{R}$ satisfies the following conditions
   (i) For $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
   (ii) $d(x, y) \leq d(z, x) + d(z, y)$ for all $x, y, z \in X$.
   Show that $d$ is a metric on $X$.

6. Let $X$ be a nonempty set. A function $\rho : X \times X \rightarrow \mathbb{R}$ satisfies the following conditions
   (i) For $x, y \in X$, $\rho(x, y) = 0$ if and only if $x = y$.
   (ii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.
   Show that these conditions are not sufficient to make the function a metric on $X$.

7. Can $d_1, d_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ where $d_1(x, y) = \exp(|x - y|)$, $d_2(x, y) = \max\{x - y, 0\}$ be metrics on $\mathbb{R}$?

8. Show that the set $\mathbb{Q}$ of all rational numbers is a metric space when equipped with the metric $d$, such that $d(u, v) = |u - v|$. Show that it is not complete.

9. Consider $X = C([-1, 1]; \mathbb{R})$ the space of all real-valued continuous functions on $[-1, 1]$. Define a function $d_2 : X \times X \rightarrow \mathbb{R}$ by $d_2(f, g) = \left( \int_{-1}^{1} |f(x) - g(x)|^2 \right)^{1/2}$. Show $d_2$ is a metric on $X$. By studying the sequence of functions $f_n$, given by $f_n(x) = \frac{1}{2} + \frac{1}{\pi \arctan nx}$, $-1 \leq x \leq 1$, show that the space $(X, d_2)$ is not complete.

10. Consider $f : [0, +\infty) \rightarrow [0, +\infty)$ defined by $f(u) = u + e^{-u}$. Show that for $u \neq v$, $d(f(u), f(v)) < d(u, v)$, yet $f$ has no fixed point.