MATH 102 - Quiz 4 Section number: Student ID:

Instructions: You are required to attempt all questions. Each is worth 5 points. Answers with insufficient working will result in zero for that question.

1. Examine the series $\sum_{n=0}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ for conditional or absolute convergence. Show how you arrived at your conclusion.

Solution:
Let $a_n = (-1)^{n-1}\frac{n}{n^2+1}$. First, we consider $\sum_{n=0}^{\infty}|a_n|$.

Employ limit comparison test with the series $\sum_{n=0}^{\infty}\frac{1}{n}$. So, $\lim_{n\to\infty}\frac{n}{n^2+1}\div\frac{1}{n} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$ ($\neq 0$ / $\neq \infty$). So, the limit comparison test tells us that the two series $\sum_{n=0}^{\infty}|a_n|$ and $\sum_{n=0}^{\infty}\frac{1}{n}$ have the same characteristics. As $\sum_{n=0}^{\infty}\frac{1}{n}$ is a harmonic series (which diverges by p-series test), $\sum_{n=0}^{\infty}|a_n|$ diverges.

Second, we apply the alternating series test for $\sum_{n=0}^{\infty}a_n$ which requires that $\frac{n}{n^2+1}$ be a decreasing sequence and that $\lim_{n\to\infty}\frac{n}{n^2+1} = 0$. As both conditions are satisfied, the alternating series converges. Therefore, we have conditional convergence.

2. Determine whether $\sum_{n=1}^{\infty}(1+\frac{1}{n})^2e^{-n}$ converges or diverges.

Solution:
Let $a_n = (1+\frac{1}{n})^2e^{-n}$ and $b_n = e^{-n}$. Using the limit comparison test, we have $\lim_{n\to\infty}\frac{a_n}{b_n} = (1+1/n)^2 = 1 > 0$. By the limit comparison test, $\sum_{n=0}^{\infty}a_n$ converges.

3. Find the interval of convergence of $1 + \frac{(x-3)}{1^2} + \frac{(x-3)^2}{2^2} + \frac{(x-3)^3}{3^2} + \ldots + \frac{(x-3)^{n-1}}{(n-1)^2} + \ldots$

Solution:
We apply the ratio test to the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2}$ and we require $\lim_{n\to\infty} |\frac{(x-3)^{n+1}}{(n+1)^2} \frac{n^2}{(x-3)^n}| = |x-3| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = |x-3| < 1$ for convergence. Thus we at least know that we have convergence for $2 < x < 4$.

Now, we look at the boundary values: $x = 2$ and $x = 4$. At $x = 2$, we obtain the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$ which satisfies the two criterion for a convergent alternating series: (a) $\frac{1}{n^2}$ is a decreasing function of n and $\lim_{n\to\infty} \frac{1}{n^2} = 0$. At $x = 4$, we obtain the series $\sum_{n=0}^{\infty} \frac{1}{n^2}$ which converges by the p-series test.

Thus, the interval of convergence is $2 \le x \le 4$. The presence of 1 does not change the interval of convergence.