1. Sketch the region between the surface \( y = \sqrt{x^2 + z^2} \) and \( x^2 + z^2 = 1 \) for \( 1 \leq y \leq 2 \).

2. For the function \( f(x, y) = \sin(x^2 + y) \), compute the linear approximation of \( f(1.02, -1.01) \).

\[
\frac{\partial f}{\partial x} (1, -1) = \sin(1) = 0 \\
\frac{\partial f}{\partial y} (1, -1) = 2x \cos(x^2 + y) \bigg|_{x=1, y=-1} = 2 \\
\frac{\partial f}{\partial y} (1, -1) = \cos(x^2 + y) \bigg|_{x=1, y=-1} = 1 \\

\begin{align*}
\Delta x &= 1.02 - 1 = 0.02 \\
\Delta y &= -1.01 - (-1) = -0.01 \\
L(x, y) &= f(1, -1) + \frac{\partial f}{\partial x} (1, -1) \Delta x + \frac{\partial f}{\partial y} (1, -1) \Delta y \\
L(1.02, -1.01) &= 0 + 2(0.02) + 1(-0.01) = 0.03
\end{align*}

\[ -1 \]
3. Transform the equation \( \rho \sin \phi - \cos \theta = 0 \) from spherical into cylindrical coordinates, and sketch the surface.

\[
\begin{align*}
\text{(a) } & \quad \rho \sin \phi = \kappa, \\
\text{(b) } & \quad \cos \theta = \cos \theta
\end{align*}
\]

It is a cylinder.

\[
\begin{align*}
\kappa &= r \cos \theta \\
\rho^2 &= \kappa \csc \phi \\
x^2 + y^2 &= \frac{r^2}{\kappa} \\
(x - \frac{1}{2})^2 + \frac{y^2}{(\frac{1}{2})^2} &= \frac{1}{\kappa}
\end{align*}
\]

4. Name the boundary surfaces, and sketch the solid given in cylindrical coordinates by the inequalities \( 1 + r^2 \leq z^2 \leq 4 - z^2 \).

\[
\begin{align*}
1 + r^2 &= z^2 & \rightarrow & \quad x^2 + y^2 = z^2 - 1 & \text{It is a 2-sheeted hyperboloid} \\
h - r^2 &= z^2 & \rightarrow & \quad x^2 + y^2 + z^2 = 4 & \text{It is a sphere.}
\end{align*}
\]
5. Consider the function \( f(x, y) = \sqrt{x^2 + y^2 - 1} \):

(a) Describe in words and sketch the domain of \( f \).

\[
\text{Domain } (f) = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1 \}
\]

are all the points outside and on the \textit{unit circle}.

(b) Describe in words and sketch the graph of \( f \).

\[
\text{Graph } (f) = \{ (x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{x^2 + y^2 - 1} \}
\]

= \[ \{ (x, y, z) \in \mathbb{R}^3 \mid z \geq 0, x^2 + y^2 = z^2 + 1 \} \]

\( \text{It is the upper - half of a } \frac{1}{4} \text{- \textit{cylindrical hyperboloid.}} \)
6. For the function \( f(x, y) = \tan(xy) \), compute the expression \( f_y + f_{xy} \).

\[
\frac{\partial f}{\partial y} (x, y) = \frac{z}{\cos^2(xy)}
\]

\[
\frac{\partial f}{\partial x} (x, y) = \frac{1}{\cos^2(xy)} + x \cdot \frac{2 \sin(xy) \cdot y}{\cos^3(xy)}
\]

\[
f_y + f_{xy} = \frac{z}{\cos^2(xy)} + \frac{1}{\cos^2(xy)} + \frac{2xy \cdot \sin(xy)}{\cos^3(xy)}
\]

7. Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) for the function \( z \) which is defined implicitly by the equation

\[\tan^{-1}(y z) = z - x = 0.\]

\[
\frac{\partial z}{\partial x} = - \frac{f_x (x, y, z)}{f_z (x, y, z)} = - \frac{1}{z - x} = \frac{1}{1 + y^2 z^2} \cdot \frac{y^2}{1 + y^2 z^2}
\]

\[
\frac{\partial z}{\partial y} = - \frac{f_y (x, y, z)}{f_z (x, y, z)} = - \frac{1}{z - x} = \frac{1}{1 + y^2 z^2} \cdot \frac{y}{1 + y^2 z^2}
\]
8. In the following problems: if the limit exists, compute it; if the limit does not exist, prove it.

(a) \[ \lim_{(x,y,z) \to (0,0,0)} \frac{x^2 + y^2 + z^2}{\sqrt{4 + x^2 + y^2 + z^2}} = \frac{\sqrt{4 + x^2 + y^2 + z^2}}{4 + x^2 + y^2 + z^2} = 1 \]

(b) \[ \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + 3y^2} \]

\[ \text{Approach } (0,0) \text{ along } \gamma = \gamma_1 \text{ (or } y = 0) \] \[ \lim_{(x,0) \to (0,0)} \frac{x \cdot 0}{x^2 + 3 \cdot 0^2} = 0 \]

\[ \text{Approach } (0,0) \text{ along } \gamma = y^4 \text{ (or } x = \text{ const.}, y^4) \]

\[ \lim_{(y^4, y) \to (0,0)} \frac{y^4 \cdot y}{y^8 + 4y^2} = \frac{y^5}{y^8 + 4y^2} \]

The limit does not exist.
9. Find the equation of the tangent plane and the normal line to the surface \( z = e^x - y^2 \) at the point \((1, -1, 1)\).

\[
\nabla z = \left\langle e^x - y^2, -2y e^x - y^2 \right\rangle
\]

\[
\nabla z (1, -1) = \left\langle 1, 2 \right\rangle
\]

The equation of the tangent plane is:

\[
\begin{align*}
\hat{z} - 1 &= (x - 1) + 2(y + 1) \\
\Rightarrow \quad \hat{z} &= x + 2y + 2
\end{align*}
\]

The parametric equation of the normal line is:

\[
\begin{align*}
\hat{x} &= 1 + t \\
\hat{y} &= -1 + 2t \\
\hat{z} &= 1 - t
\end{align*}
\]

10. Given the surface defined implicitly by the equation \( ye^{2z} + 2x + z = 3 \), and the point \((1, 1, 0)\) on it, compute the directional derivative of \( z \) in the direction \( u = (3, 4) \).

\[
\hat{f}(x, y, z) = \frac{\partial f}{\partial x} = ye^{2z} + 2x + z
\]

\[
\frac{\partial z}{\partial x} = -\frac{f_x(x, y, z)}{f_z(x, y, z)} = -\frac{y e^{2z} + 2}{x y e^{2z} + 1},
\]

\[
\Rightarrow \quad \frac{\partial z}{\partial x} (1, 1, 0) = -\frac{2}{2} = -1
\]

\[
\frac{\partial z}{\partial y} = -\frac{f_y(x, y, z)}{f_z(x, y, z)} = -\frac{e^{2z}}{x y e^{2z} + 1},
\]

\[
\Rightarrow \quad \frac{\partial z}{\partial y} (1, 1, 0) = -\frac{1}{2}
\]

normalized \( u \): \( \hat{u} = \frac{u}{||u||} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \)

\[
\nabla z (1, 1) = \frac{\partial}{\partial x} e^{2z} (1, 1) = 2 \cdot \frac{e^{2z}}{5} (1, 1) + \frac{e^{2z}}{5} (-1) = \frac{2}{5} \cdot (1) + \frac{2}{5} \cdot (-1) = -1
\]
11. Use the chain rule to find the partial derivative \( \frac{\partial Y}{\partial t} \) for \( Y = w \cdot \tan^{-1}(uv) \), where \( u = r + s \), \( v = s + t \), and \( w = t + r \).

\[
\frac{\partial Y}{\partial t} = \frac{\partial Y}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial Y}{\partial v} \cdot \frac{\partial v}{\partial t} + \frac{\partial Y}{\partial w} \cdot \frac{\partial w}{\partial t}
\]

\[
= \frac{w \cdot y}{1 + u^2 v^2} \cdot 0 + \frac{w \cdot u}{1 + u^2 v^2} \cdot 1 + \tan^{-1}(uv) \cdot \frac{1}{v}
\]

\[
= \frac{(t + r)(u + v)}{a + (u + v)^2 \cdot (u + v)^2} + \tan^{-1}\left((u + v)(n + t)\right)
\]

12. Find the direction in which the function \( f(x, y) = x^2 + xy + y^2 \) increases the most rapidly at \((-1, 1)\). Find the derivative of \( f \) in this direction.

\[
\nabla f(x, y) = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right> = \left< 2x + y, \frac{\partial f}{\partial y} \right>
\]

\[
\nabla f(-1, 1) = \left< -1, 1 \right>
\]

Normalize \( \nabla f \) \( \rightarrow \nabla f = \left< -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right> \)

\[
D_u f(-1, 1) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}
\]