Key Major I, Math 430

1. Use the formula
$$1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z}$$
with $z = e^{i\theta}$ to show that
$$\frac{|\sin n\theta/2|}{|\sin \theta/2|} \leq n, \quad \theta \neq 0, \pm 2\pi, \pm 4\pi, \ldots$$

Solution: $1 + e^{i\theta} + \cdots + e^{i(n-1)\theta} = \dfrac{1 - e^{in\theta}}{1 - e^{i\theta}}$. So $\dfrac{|1 - e^{in\theta}|}{|1 - e^{i\theta|}} = |1 + e^{i\theta} + \cdots + e^{i(n-1,\theta)}| \leq 1 + |e^{i\theta}| + \cdots + |e^{i(n-1)\theta}| \leq n.$ (*).
Now
$$\begin{aligned} |1 - e^{i\theta}| &= ((1 - e^{i\theta})(1 - e^{-i\theta}))^{1/2} \\ &= (1 - e^{-i\theta} - e^{+i\theta} + 1)^{1/2} \\ &= (2 - 2\cos\theta)^{1/2} \\ &= (2(2\sin^2\theta/2))^{1/2} = 2|\sin\theta/2| \end{aligned}$$

Similarly,
$$|1 - e^{-in\theta}| = 2|\sin n\theta/2|.$$

Therefore, from (*)
$$\frac{|\sin n\theta/2|}{|\sin \theta/2|} \leq n, \quad \theta \neq 0, \pm 2\pi, \pm 4\pi, \ldots$$

2. (a) $z^4 + z^2 + 1 = 0$.
Put $z = w$.
Get $w^2 + w + 1 = 0$. So, $w = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2}$. So, $|w| = 1$ and $\arg w$ is given by $\tan\theta \pm \sqrt{3}$. Since w is the 2nd or 3rd quadrant, $\arg w = -\pi/3 + \pi = 2\pi/3, \pi/3 + \pi = 4\pi/3$. So, $w = e^{i2\pi/3}, e^{i4\pi/3}$. Hence $z = \sqrt{w} = \pm e^{i\pi/3}, \pm e^{i2\pi/3}$.

(b) $(z+1)^7 = z^7 \Rightarrow z \neq 0$, so z is a root of $\left(\dfrac{z+1}{z}\right)^7 = 1$.
So $\dfrac{z+1}{z} = e^{\frac{2\pi i}{7}j}$, $\quad j = 0, 1, 2, 3, 4, 5, 6$ are the possible solutions. Put $w = e^{\frac{2\pi i}{7}}$.
Now $\dfrac{z+1}{z} = w^j \Rightarrow z + 1 = w^j z$. So, $z(1 - w^j) = -1$(*) and $z = \dfrac{1}{w^j - 1}$: $\quad j = 0$ is ruled out, because of (*). So, the roots are $z = \dfrac{1}{w^j - 1}$, $\quad j = 1, 2, \ldots, 6$.

(c) If $w = \dfrac{1}{z}$ with $|z - 1| = 1$, then for $z \neq 0$, $\left|\dfrac{1}{w} - 1\right| = 1$. So, $\left(\dfrac{1}{w} - 1\right)\left(\dfrac{1}{\overline{w}} - 1\right) = 1$
$$\begin{aligned} (1 - w)(1 - \overline{w}) &= w\overline{w} \\ 1 - w - \overline{w} + w\overline{w} &= w\overline{w} \end{aligned}$$

Or: $1 = w + \overline{w}$. Thus, the image of the circle (without 0) is the line re $w = \dfrac{1}{2}$.

(d) If $w = \dfrac{1}{z}$ with $|z-1| = 2$, then in this case w is defined for all z on the circle $|z-1| = 2$ as 0 is not on the circle. Therefore w satisfies the equation

$$\begin{aligned}
\left|\frac{1}{w} - 1\right| &= 2 \\
\left(\frac{1}{w} - 1\right)\left(\frac{1}{\overline{w}} - 1\right) &= 4 \\
(1-w)(1-\overline{w}) &= 4w\overline{w} \\
1 - w - \overline{w} + w\overline{w} &= 4w\overline{w}
\end{aligned}$$

or $3w\overline{w} + w + \overline{w} - 1 = 0$. If $w = (x+iy)$, then w satisfies the equation

$$3(x^2+y^2) + 2x - 1 = 0$$

which is a circle:

$$\begin{aligned}
&x^2 + y^2 + \frac{2}{3}x - \frac{1}{3} = 0 \\
&x^2 + \frac{2}{3}x + \frac{1}{9} + y^2 - \frac{1}{3} - \frac{1}{9} = 0 \\
&\left(x + \tfrac{1}{3}\right)^2 + y^2 = \tfrac{1}{3} + \tfrac{1}{9} = \tfrac{4}{9}
\end{aligned}$$

So, it is a circle centered at $\left(-\dfrac{1}{3}, 0\right)$ of radius $\dfrac{2}{3}$.

3. (a) The line through $N(0,0,1)$ and $(x,y,0)$ has direction vector $[x,y,-1]$. So, its equations are

$$X = tx, Y = ty, Z = 1-t.$$

This line intersects the unit sphere at points given by

$$\begin{aligned}
&t^2x^2 + t^2y^2 + (1-t)^2 = 1 \\
&t^2x^2 + t^2y^2 + 1 - 2t + t^2 = 1 \\
&t^2x^2 + t^2y^2 + t^2 - 2t = 0 \\
&t(tx^2 + ty^2 + t - 2) = 0.
\end{aligned}$$

So, $t = 0$ or $t = \dfrac{2}{1+x^2+y^2}$. So, the points of intersection are

$$\begin{aligned}
&(0,0,1) \text{ and } \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 1 - \frac{2}{1+x^2+y^2}\right) \\
&= \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right).
\end{aligned}$$

The function

$$(x,y) \mapsto \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 1 - \frac{2}{1+x^2+y^2}\right)$$

is 1 : 1 and onto the shere \(0, 0, 1). The circle $|z| = 3$ is mapped to points whose Z-coordinate is

$$1 - \frac{z}{1+9} = \frac{8}{10} = \frac{4}{5}.$$

The intersection of a plane with a sphere (if it intersects it) is a circle. So, the intersection of the plane $Z = \frac{4}{5}$ with the sphere is the image of $|z| = 3$.

(b) The circle $|z| = r$ is the intersection of the plane

$$Z = \frac{r^2 - 1}{r^2 + 1} = 1 - \frac{2}{(r^2 + 1)}.$$

The function $r \to 1 - \frac{2}{(r^2+1)}$ is an increasing function of r as $0 < r < s \to r^2 < s^2$ and $\frac{1}{r^2+1} > \frac{1}{s^2+1}$ so $\frac{-2}{1+r^2} < \frac{-2}{1+s^2}$. When $r \to \infty, 1 - \frac{2}{(1+r^2)} \to 1$. So, the image of the set $|z| > 3$ is the upper cap cut by the plane $Z = \frac{4}{5}$ on the unit sphere.

4. $f(z) = (x^{4/3}y^{5/3} + ix^{5/3}y^{4/3})\frac{1}{x^2 + y}$ if $z \neq 0$

$f(0) = 0$.

Show: CR equations hold at 0, but the function is not differentiable at 0.

Solution:

$$\begin{aligned}
\text{Re } f(z) &= \frac{x^{4/3}y^{5/3}}{x^2 + y^2} = u(x, y) \\
\text{Im } f(z) &= \frac{x^{5/3}y^{4/3}}{x^2 + y^2} = v(x, y) \\
\frac{\partial u}{\partial x}(0, 0) &= \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} \\
&= \lim_{x \to 0} \frac{0}{x} = 0 \\
\frac{\partial u}{\partial y}(0, 0) &= \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = 0 \\
\frac{\partial v}{\partial x}(0, 0) & \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = 0 \\
\frac{\partial v}{\partial y}(0, 0) & \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = 0.
\end{aligned}$$

Now,

$$\begin{aligned}
f'(0) &= \lim_{z \to 0} \frac{f(z) - f(0)}{z} \\
&= \lim_{z \to 0} \frac{f(z)}{z}.
\end{aligned}$$

This should be independent of how we approach 0 – if $f'(0)$ exists.

Along the x-axis, $z = x + i0$. So,

$$\lim_{\substack{z\to 0\\ \text{along} x-\text{axis}}} \frac{f(z)}{z} = \lim_{x\to 0} \frac{f(x+i0)}{x}$$

$$= \lim_{x\to 0} \frac{0}{x^2}\frac{1}{x} = 0$$

$$\lim_{\substack{z\to 0\\ z=x+ix}} = \lim_{x\to 0} \frac{x^{4/3}x^{5/3} + ix^{5/3}x^{4/3}}{x^2+x^2}\frac{1}{x+ix}$$

$$= \lim_{x\to 0} \frac{x^3+ix^3}{2x^2x(1+i)} = \lim_{x\to 0} \frac{1+i}{2(1+i)} = \frac{1}{2}$$

So $f'(0)$ does not exist.

5. (a) Suppose $f(z)$ is analytic and non-zero in a domain D. Take a point z_0 in D. So, $f(z_0) \neq 0$. Take a branch of log defined near $f(z_0)$. Therefore, near z_0, the function $\log f(z)$ is analytic and its real part is $\log|f(z)|$. Therefore $\log|f(z)|$ is harmonic and so $2\log|f(z)|$ is harmonic.

   (You can do this also by direct computations:

   $$\log|f(z)|^2 = \log(u^2+v^2)$$

   Use CR equations to prove that $\dfrac{\partial^2}{\partial x^2}\log(u^2+v^2) + \dfrac{\partial^2}{\partial y^2}\log(u^2+v^2) = 0$.

   (b) The function

   $$\log\frac{|z|}{3} = \frac{\log(x^2+y^2)^{1/2}}{3}$$

   is harmonic and when $|z| = 3$, it is equal to $\log 1 = 0$ (here log is w.r.t base e).