King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 430 Final Exam
First Semester 2009–2010(092)
March 31, 2010

Name: ______________________________

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1. Find the residue at all singularities of $f(z) = \dfrac{e^z}{\sin^2(z)}$.

2. (a) Let $f(\theta) = |e^{i\theta\epsilon_1}a + e^{i\theta\epsilon_2}b|, \quad -\infty < \theta < \infty$, where $\epsilon_1, \epsilon_2 \in \mathbb{R}$. Show that if $\epsilon_1 \neq \epsilon_2$, then the maximum value of $f(\theta) = |a| + |b|$.

   (b) Show that if $a, b \in \mathbb{C}$ and $|a| < 1,\ |b| < 1$, then $\dfrac{|a-b|}{|1-\overline{a}\,b|} < 1$.

3. Evaluate the integrals by any method

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} \quad a, b > 0.$

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x)^n}$.

4. Let $f(z)$ be an analytic function with Taylor series expression $f(z) = \displaystyle\sum_{n=0}^{\infty} c_n(z-a)^n$.

   Show

   (a) $\operatorname{Re} f(z) = \dfrac{1}{2}\left(\displaystyle\sum_{n=0}^{\infty} c_n(x+iy-a)^n + \sum_{n=0}^{\infty} \overline{c}_n(x-iy-\overline{a})^n\right) = u(x,y).$ here: $z = x + iy$.

   (b) Let $U(z_1, z_2) = \dfrac{1}{2}\left(\displaystyle\sum_{n=0}^{\infty} c_n(z_1+iz_2-a)^n\right) + \dfrac{1}{2}\left(\displaystyle\sum_{n=0}^{\infty} \overline{c}_n(z_1-iz_2-\overline{a})^n\right)$. Explain why $U(z_1, z_2)$ is analytic in both the variables z_1, z_2.

   (c) Show that $U\left(\dfrac{z+a}{2}, \dfrac{z-a}{2i}\right) = \dfrac{1}{2}f(z) + \dfrac{1}{2}\overline{f(\overline{a})}.$

   (d) Using (c), find a holomorphic function whose real part is $u(x,y) = e^{\left(\frac{x}{x^2+y^2}\right)} \cos\left(\dfrac{y}{x^2+y^2}\right)$.

5. Prove the following by using steps a), b), c).

   If f is analytic in the unit disc $|z| < 1$ and satisfies the conditions $f(0) = 0$ and $|f(z)| \leq 1$ for all z in U, then $|f(z)| \leq |z|$ for all z in U.

   (a) Define $F(z) = \dfrac{f(z)}{z}$, for $z \neq 0$, and $F(0) = f'(0)$. Show that F is analytic in U.

   (b) Let $\zeta \neq 0$ be any fixed point in U and let r be any real anumber that satisfies $|\zeta| < r < 1$. Show by means of the maximum-modulus principle that if C_r denotes the circle $|z| = r$, then $|F(\zeta)| \leq \max_{z \text{ on } C_r} \dfrac{|f(z)|}{r} \leq \dfrac{1}{r}$.

   (c) Letting $r \to 1$ from below, deduce that $|f(\zeta)| \leq |\zeta|$ for all ζ in U.