1. [8pts] Let \( A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 6 & -4 & 5 & -3 \\ 8 & -4 & 1 & 0 \\ 4 & -1 & 0 & 7 \end{bmatrix} \).

(a) Find the LU decomposition of \( A \).

(b) Use (a) to solve the system \( Ax = b \), where \( b = [1 \quad 2 \quad 2 \quad 1]^T \).
2. [4pts] Suppose that $A$ is an invertible symmetric matrix with $A = LDU$ (its LDU factorization). Show that $U = L^T$. 
3. [6pts] Find a basis for the column space and a basis for the left nullspace of the matrix

\[ A = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 4 & 6 \\
0 & 0 & 0 & 1 & 2 \\
\end{bmatrix} \]
4. [6pts] (a) Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the composition: Reflection in the line $y = x$, followed by counterclockwise rotation through $30^\circ$, followed by reflection in the line $y = -x$. Find the matrix $A$ of $L$ in the standard basis of $\mathbb{R}^2$.

(b) Let $f : V \rightarrow V$ be a linear operator on a vector space $V$. Show that:

(i) $\ker f \subseteq \ker(f \circ f)$.

(ii) $\text{Im}(f \circ f) \subseteq \text{Im} f$. Deduce from this that $\text{Rank}(A^2) \leq \text{Rank}(A)$ for any matrix $A$. 
5. [8pts] (a) Let $B$ be an $m \times n$ edge-node incidence matrix of a simple connected directed graph. Let $s_j$ be the sum of entries in the $j$th column of $B$. What does $s_1$ represent? What is the value of $\sum_{j=1}^n s_j$?

(b) Let $G$ be the directed graph below.
(i) Write down the edge-node incidence matrix $A$ of the graph
(ii) Find conditions on $b_1, b_2, b_3, b_4, b_5$ for which the system $Ax = (b_1, b_2, b_3, b_4, b_5)^T$ has at least one solution.
(iii) What is the number of loops in the graph $G$?