1. [20pts] (a) Find the norm and the condition number of the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. What is the condition number of the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$?

(b) Consider the system $Ax = b$ where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$. Suppose we perturb $b$ so that it changes to $b' = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$.

(i) Without solving $Ax = b$ or $Ax = b'$, give an upper bound on the relative error $\frac{\|\delta x\|}{\|x\|}$.

(ii) Solve the systems $Ax = b$ and $Ax = b'$ to determine the exact relative error.

(c) (i) Use the definition of the norm to prove that for all $n \times n$ matrices $A, B$ and all vectors $v \in \mathbb{R}^n$, $\|ABv\| \leq \|A\| \|B\| \|v\|$ and that $\|AB\| \leq \|A\| \|B\|$.

(ii) Show, that for all nonnegative integers $n$, we have $\text{cond}(A^n) \leq (\text{cond}(A))^n$. 
2. [20pts] Let \( A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \).

(a) Find a singular value decomposition of \( A \).

(b) Find a basis for the left nullspace of \( A \).

(c) Find the pseudoinverse \( A^+ \) of \( A \).

(d) Find the minimum length least squares solution of the equation \( Ax = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \).

(e) Show that if \( M \) is an \( m \times n \) matrix and \( Q \) is an orthogonal \( m \times m \) matrix then \( M \) and \(QM\) have the same singular values.
3. [20pts] (a) State Gerschgorin theorem. Let $A$ be a real square matrix with all entries nonnegative and such that the sum of the entries in each row is equal to 1. Show that if $\lambda$ is an eigenvalue of $A$ then $|\lambda| \leq 1$.

(b) Without computing the characteristic equation, use Gerschgorin theorem to show that all the eigenvalues of the matrix

$$\begin{bmatrix}
0 & 1 & 0 & 0 \\
2 & 5 & 0 & 0 \\
1/2 & 0 & 3 & 1/2 \\
0 & 0 & 3/4 & 7
\end{bmatrix}$$

are real.
4. [20pts] (a) Let $A = [a_{ij}]_{1 \leq i,j \leq n}$ be a real symmetric matrix such that $a_{11} = a_{12} = a_{21} = a_{22} = 0$. Compute the Rayleigh quotient $R(x)$ for $A$ at $x = [1 \ 1 \ 0 \ \cdots \ 0]^T$. What can you deduce about the sign of the largest eigenvalue $\lambda_{\text{max}}$ and the sign of the smallest eigenvalue $\lambda_{\text{min}}$ of $A$?

(b) Let $A$ be an $n \times n$ symmetric positive definite matrix, $b \in \mathbb{R}^n$ and $P(x) = \frac{1}{2} x^T A x - x^T b$. For which $x$ does $P(x)$ reach its minimum value? What is this value?

(c) Find the minimum value of $x_1^2 + 4x_1x_2 + 8x_2^2 - 12x_2$. 
5. [20pts] Label each of the following statements as **True** or **False**.

(a) If \( \mathbf{A} \) is a real symmetric matrix such that all its eigenvalues are equal to 1, then \( \mathbf{A} \) is the identity matrix.

(b) If \( \mathbf{A} \) and \( \mathbf{B} \) are real matrices that have the same characteristic polynomials, then \( \mathbf{A} = \mathbf{B} \).

(c) If \( \mathbf{Q} \) is a \( 3 \times 2 \) matrix with orthonormal columns then \( \| \mathbf{Qx} \| = \| \mathbf{x} \| \) for all \( \mathbf{x} \in \mathbb{R}^2 \).

(d) If \( \mathbf{P} \) is a projection matrix, then \( c \mathbf{P} \) is a projection matrix for each scalar \( c \).

(e) If \( \mathcal{V} \) is the subspace spanned by \((1, 1, 1)\) and \((0, 0, 1, 0)\), then the vector in \( \mathcal{V} \) closest to the vector \( \mathbf{b} = (0, 1, 0, -1) \) in \( \mathcal{V}^\perp \) is \((0, 0, 0, 0)\).