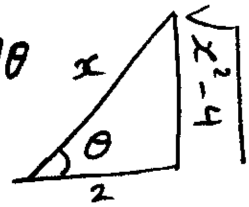
KFUPM Summer 093 Name: Solutions Serial #: ______

MATH 102-4-6 Quiz 2 (Part II) ID: #: ______ Sec. #: ______

1. (5-points) Evaluate $\int \frac{\sqrt{x^2-4}}{x^3}\,dx. = I$

Let $x = 2\sec\theta \Rightarrow \sqrt{x^2-4} = 2\tan\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$

$$I = \int \frac{2\tan\theta}{8\sec^3\theta}\,2\sec\theta\tan\theta\,d\theta = \frac{1}{2}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$



$$= \frac{1}{2}\int \sin^2\theta\,d\theta = \frac{1}{4}\int (1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C = \frac{1}{4}\left[\theta - \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{4}\left[\sec^{-1}\left(\frac{x}{2}\right) - \frac{\sqrt{x^2-4}}{x}\cdot\frac{2}{x}\right] + C$$

$$= \frac{1}{4}\sec^{-1}\left(\frac{x}{2}\right) - \frac{\sqrt{x^2-4}}{2x} + C$$

2. (5-points) Determine whether the integral $\int_1^{11/3} \frac{1}{(3x-11)^{1/3}}\,dx$ is convergent or divergent. Find its value if it is convergent.

The integrand has infinite discontinuity at $\frac{11}{3} \in [1, \frac{11}{3}]$

$$\Rightarrow \int_1^{11/3} \frac{1}{(3x-11)^{1/3}}\,dx = \lim_{t\to(\frac{11}{3})^-} \int_1^t (3x-11)^{-1/3}\,dx$$

$$= \lim_{t\to(\frac{11}{3})^-} \left[\frac{1}{3}\cdot\frac{3}{2}(3x-11)^{2/3}\right]_1^t$$

$$= \frac{1}{2}\lim_{t\to(\frac{11}{3})^-} \left[(3t-11)^{2/3} - (-8)^{2/3}\right] = \frac{1}{2}[0-4]$$

$$= -2.$$

3. (5-points) Evaluate $\int \frac{3x^2+13x+24}{x^3+4x^2+8x}\,dx.$ $= I$

$$\frac{3x^2+13x+24}{x^3+4x^2+8x} = \frac{3x^2+13x+24}{x(\underbrace{x^2+4x+8}_{\text{irreducible}})} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+8}$$

$$\Rightarrow \quad 3x^2+13x+24 = A(x^2+4x+8) + (Bx+C)\,x$$

$\boxed{x=0}$ $\quad 24 = 8A \Longrightarrow \boxed{A=3}$

$\boxed{\text{coef } x^2}$ $\quad 3 = A+B \Longrightarrow \boxed{B=0}$

$\boxed{\text{Coef } x}$ $\quad 13 = 4A + C \Longrightarrow \boxed{C=1}$

$$I = \int \frac{3}{x}\,dx + \int \frac{1}{x^2+4x+8}\,dx$$

$$= 3\ln|x| + \int \frac{1}{(x+2)^2+4}\,dx$$

$$= 3\ln|x| + \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C.$$

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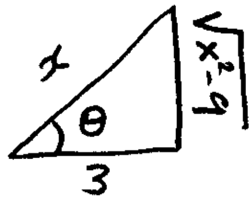
1. (5-points) Evaluate $\int \frac{\sqrt{x^2-9}}{x^3}\,dx. = I$

Let $x = 3\sec\theta \Rightarrow \sqrt{x^2-9} = 3\tan\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$

$$I = \int \frac{3\tan\theta}{27\sec^3\theta}\, 3\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$



$$= \frac{1}{3}\int \sin^2\theta\,d\theta = \frac{1}{6}\int (1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{6}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{1}{6}\left[\theta - \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{6}\left[\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x}\right] + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

2. (5-points) Determine whether the integral $\int_{1/2}^{9/2} \frac{1}{(2x-9)^{1/3}}\,dx$ is convergent or divergent. Find its value if it is convergent.

The integrand has an infinite discontinuity at $x = \frac{9}{2} \in [\frac{1}{2}, \frac{9}{2}]$

$$\Rightarrow \int_{1/2}^{9/2} \frac{1}{(2x-9)^{1/3}}\,dx = \lim_{t\to(\frac{9}{2})^-} \int_{1/2}^{t} (2x-9)^{-1/3}\,dx$$

$$= \lim_{t\to(\frac{9}{2})^-} \left[\frac{1}{2}\cdot\frac{3}{2}(2x-9)^{2/3}\right]_{1/2}^{t}$$

$$= \frac{3}{4}\lim_{t\to(\frac{9}{2})^-}\left[(2t-9)^{2/3} - (-8)^{2/3}\right] = \frac{3}{4}\left[0-4\right]$$

$$= -3.$$

3. (5-points) Evaluate $\int \frac{x^2+9x+8}{x^3+4x^2+8x}\,dx.$ $= I$

$$\frac{x^2+9x+8}{x^3+4x^2+8x} = \frac{x^2+9x+8}{x(\underbrace{x^2+4x+8}_{\text{irreducible}})} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+8} \Rightarrow$$

$$x^2+9x+8 = A(x^2+4x+8) + (Bx+C)x$$

$\boxed{x=0} \Rightarrow \quad 8 = 8A \Rightarrow \boxed{A=1}$

$\boxed{\text{coef } x^2} \Rightarrow \quad 1 = A+B \Rightarrow \boxed{B=0}$

$\boxed{\text{coef } x} \quad 9 = 4A+C \Rightarrow \boxed{C=5}$

$$\Rightarrow \quad I = \int \frac{1}{x}\,dx + \int \frac{5}{x^2+4x+8}\,dx$$

$$= \ln|x| + 5\int \frac{1}{(x+2)^2+4}\,dx$$

$$= \ln|x| + \frac{5}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C.$$