Answer the questions in the space provided. You must show your work or explain your solution otherwise points may be deducted. If you make an unnecessary approximation in your solution to a problem, your answer will be judged on its accuracy. Points may be deducted for poor or inappropriate approximation.

1. Write clearly.
2. Show all your steps.
3. No credits will be given to wrong steps.
4. Calculators and mobile phones are NOT allowed in this exam.

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Problem 1 (14 pts).

a) Find the rectangular (Cartesian) equation of the parametric curve given by

\[ x = t - \frac{\sqrt{t}}{2}, \quad \text{and} \quad y = t + \frac{\sqrt{t}}{2} \]

We have \( y + x = 2t \), and \( y - x = \frac{3\sqrt{t}}{2} \), \[ t = \frac{y - x}{3} \]

Now using \( t = \frac{y + x}{2} \), we get \( y - x = \frac{y + x}{2} \), \( or \)

\[ (y - x) = \frac{y + x}{2} \]

or \( 2(y - x) = y + x \).

b) A parametric curve is given by the equations

\[ x = \ln t \quad \text{and} \quad y = \sqrt{t}, \quad 1 \leq t \leq e^2. \]

Sketch the curve and indicate with an arrow the direction in which it is traced.

\[ x = \ln t \Rightarrow t = e^x \]

Now using \( y = \sqrt{t} \), we obtain \( y = e^{\frac{x}{2}} \), with \( 0 \leq x \leq 2 \).

The curve is given by

\[ 0 \quad 1 \quad 2 \]

\[ 1 \quad e \]

2 pts
Problem 2 (16 pts).

a) A curve $C$ is defined by the parametric equations

$$x = t - \sin t \quad \text{and} \quad y = 1 - \cos t, \quad 0 \leq t \leq 2\pi.$$  

Find (if exist) the points on $C$ where the tangent is horizontal or vertical.

$$\frac{dx}{dt} = 1 - \cos t \quad \text{and} \quad \frac{dy}{dt} = \sin t$$

$$\frac{dx}{dt} = 0 \iff t = 0 \quad \text{or} \quad t = 2\pi.$$  

$$\frac{dy}{dt} = 0 \iff t = 0 \quad \text{or} \quad t = \pi \quad \text{or} \quad t = 2\pi.$$  

Horizontal tangents:

$$\frac{dy}{dt} = 0 \quad \text{and} \quad \frac{dx}{dt} \neq 0 \implies t = \pi$$  

The corresponding point is $(\pi, 2)$.  

Vertical tangents:

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0 \implies t = 0 \quad \text{or} \quad t = 2\pi.$$  

Because we have "0" for these $t$, we use L'Hopital rule.

$$\lim_{t \to 0} \frac{dy}{dx} = \lim_{t \to 0} \frac{\sin t}{1 - \cos t} = \lim_{t \to 0} \frac{-\sin t}{\sin t} = 0$$  

and

$$\lim_{t \to 2\pi} \frac{dy}{dx} = 0.$$  

Vertical tangents occur at $t = 0$ and $t = 2\pi$.

The corresponding points are $(0, 0)$ and $(2\pi, 0)$.

b) For the curve given by

$$x = 2\cos \theta \quad \text{and} \quad y = 2\sin \theta$$

find the slope and concavity at the point $(\sqrt{2}, \sqrt{2})$.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\cos \theta}{-2\sin \theta} = -\cot \theta.$$  

Because

$$(\sqrt{2}, \sqrt{2}) \quad \text{corresponds to} \quad \theta = \frac{\pi}{4}, \quad \frac{dy}{dx}(\sqrt{2}, \sqrt{2}) = \frac{dy}{d\theta}(\theta = \frac{\pi}{4}) = -1.$$  

For the concavity we have

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(-\cot \theta)}{-2\sin \theta} = \frac{\csc \theta}{2}.$$  

$$\frac{d^2y}{dx^2}(\theta = \frac{\pi}{4}) = -\frac{1}{2} \cdot \left( \frac{3}{\sqrt{2}} \right) < 0.$$  

The curve is concave down at $\theta = \frac{\pi}{4}$, i.e. at $(\sqrt{2}, \sqrt{2})$.  

1 pt
Problem 3 (14 pts).

a) Sketch the curve with polar equation

\[ r = 2 (1 - \sin \theta) \]

\[ \theta : 0 \rightarrow \frac{\pi}{2} \rightarrow r = 2 \sqrt{2} \]
\[ \theta : \frac{\pi}{2} \rightarrow \pi \rightarrow r = 0 \]
\[ \theta : \pi \rightarrow \frac{3\pi}{2} \rightarrow r = 2 \sqrt{2} \]
\[ \theta : \frac{3\pi}{2} \rightarrow 2\pi \rightarrow r = 4 \]

b) Find the equation of the tangent line to the polar curve in part a) at \( \theta = \pi \).

\[
\text{slope} = \frac{dy}{dx} = \frac{\frac{dr}{d\theta} \cos \theta + r \sin \theta}{-r \sin \theta + \frac{dr}{d\theta} \cos \theta} \]

\[
\frac{dr}{d\theta} = -2 \cos \theta \]

\[
\frac{dy}{dx} = \frac{\frac{d}{d\theta} (1 - \sin \theta) \cos \theta - 2 \cos \theta \sin \theta}{-2 (1 - \sin \theta) \sin \theta - 2 \cos \theta}
\]

\[
\left. \frac{dy}{dx} \right|_{\theta = \pi} = \frac{-2 + 0}{-2 (1)} = 1
\]

\[
X = r \cos \theta = 2 (1 - \sin \theta) \cos \theta, \quad Y = r \sin \theta = 2 (1 - \sin \theta) \sin \theta
\]

At \( \theta = \pi \) we have the point \((-2, 0)\). The equation of the tangent line is

\[ y = (1) (x + 2) + 0 \quad \Rightarrow \quad y = x + 2 \]
Problem 4 (9 pts).

Find the area common to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 4x$.

In polar coordinates, the graph of the curves are respectively given by

$r = 2$ and $r = 4 \cos \theta$.

The area common to the curves is given by

$$A = 2 \left( A_1 + A_2 \right)$$

where

$$A_1 = \frac{1}{2} \int_{\pi/3}^{\pi} 4 \cos^2 \theta d\theta$$

and

$$A_2 = \frac{1}{2} \int_{0}^{\pi/3} 2 \theta d\theta$$

$$A = \int_{\pi/3}^{\pi} 16 \cos \theta d\theta + 4 \left( \frac{\pi}{3} \right)$$

$$= \frac{8\pi}{3} - 2\sqrt{3}$$

4 pt
Problem 5 (15pts).

a) Find the Arclength of the polar curve given by

\[ r = \sin^3 \left( \frac{\theta}{3} \right), \quad 0 \leq \theta \leq \pi. \]

\[
S = \int_0^\pi \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta
= \int_0^\pi \sqrt{\sin^6 \left( \frac{\theta}{3} \right) + \cos^2 \left( \frac{\theta}{3} \right) \sin \left( \frac{\theta}{3} \right)} \, d\theta
\]

\[
= \int_0^\pi \sqrt{\sin^4 \left( \frac{\theta}{3} \right) \left[ \sin^2 \left( \frac{\theta}{3} \right) + \cos^2 \left( \frac{\theta}{3} \right) \right]} \, d\theta
= \int_0^\pi \sin^2 \left( \frac{\theta}{3} \right) \, d\theta
\]

\[
= 3 \int_0^{\pi/3} \sin^2 u \, du
= \frac{3}{2} \int_0^{\pi/3} (1 - 2\cos u) \, du
\]

\[
= \frac{3}{2} \left[ u - \frac{1}{2} \sin 2u \right]_0^{\pi/3}
= \frac{3}{2} \left( \frac{\pi}{3} - \frac{\sqrt{3}}{4} \right),
\]

b) 

(i) Find the equation of the sphere centered at \((3, 5, -1)\) passing through the point \((4, 2, -3)\).

(ii) Find its intersection with the plane \(x - y + 2 = 0\).

i) \( r = \sqrt{(4-3)^2 + (2-5)^2 + (-3+1)^2} = \sqrt{14} \)

The equation of the sphere is

\[
(x - 3)^2 + (y - 5)^2 + (z + 1)^2 = 14 \quad (1)
\]

ii) \( x - y + 2 = 0 \) \( \Rightarrow \) \( y = x + 2 \)

Substituting \( y = x + 2 \) for \( x + 2 \) in (1) gives

\[
\left( x - 3 \right)^2 + \left( \frac{z + 1}{14} \right) = 1.
\]

This is an ellipse.
Problem 6 (15 pts).

a) Let \( \vec{a} = (1, 2, -3) \), and \( \vec{b} = (-2, -1, -5) \).

Find a vector \( \vec{v} \) with length 3 that has the same direction as \( \vec{a} - 2\vec{b} \).

\[
\vec{a} - 2\vec{b} = <1, 2, -3> - 2<-2, -1, -5> = <5, 4, 7>
\]

\( ||<5, 4, 7>|| = \sqrt{90} \). The derived vector is given by

\[
3 \cdot \frac{<5, 4, 7>}{\sqrt{90}} = \frac{3}{\sqrt{10}} <5, 4, 7> = <\frac{5}{\sqrt{10}}, \frac{4}{\sqrt{10}}, \frac{7}{\sqrt{10}}>.
\]

b) Consider the vectors \( \vec{a} = (2, 1, 2) \), and \( \vec{b} = (0, 3, 4) \).

Find the vector projection of \( \vec{a} \) onto \( \vec{b} \).

\[
\text{proj}_{\vec{b}} \vec{a} = \left( \frac{\vec{a} \cdot \vec{b}}{||\vec{b}||^2} \right) \vec{b}
\]

\[
\vec{a} \cdot \vec{b} = 2(0) + 1(3) + 2(4) = 3 + 8 = 11
\]

\( ||\vec{b}|| = \sqrt{0^2 + 3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \)

\[
\text{proj}_{\vec{b}} \vec{a} = \frac{11}{25} <0, 3, 4> = <0, \frac{33}{25}, \frac{44}{25}>
\]
Problem 7 (17 pts).

a) Find the area of the triangle with vertices \( A(1, 1, 1) \), \( B(2, -3, 2) \), and \( C(4, 1, 5) \).

\[
\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix}
1 & -1 & 1 \\
1 & -4 & 1 \\
3 & 0 & 4
\end{vmatrix} = \langle -16, -1, 12 \rangle
\]

The area of the triangle is
\[
\frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \| = \frac{1}{2} \sqrt{256 + 1 + 144} = \frac{1}{2} \sqrt{401}
\]

b) Find the component form of the vector \( \overrightarrow{u} \) that is perpendicular to the plane \( x - 3y + 4z = 0 \), and satisfying the condition \( \| \overrightarrow{u} \| = 3 \).

\( \overrightarrow{n} = \langle 1, -3, 4 \rangle \) is a normal vector to the plane. If \( \overrightarrow{u} \) is perpendicular to the plane, then

\[
\overrightarrow{u} = k \overrightarrow{n}
\]

\( k \) is a real number.

\[
\| \overrightarrow{u} \| = \| k \| \| \overrightarrow{n} \| = 3
\]

\[
\| \overrightarrow{n} \| = \sqrt{1 + 9 + 16} = \sqrt{26}
\]

\( k = \pm \frac{3}{\sqrt{26}} \) and

\[
\overrightarrow{u} = \pm \frac{3}{\sqrt{26}} <1, -3, 4> \text{ or } \overrightarrow{u} = \frac{3}{\sqrt{26}} <1, -3, 4>.
\]